



Recursive expected uncertain utility

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ABSTRACT

Gul and Pesendorfer's expected uncertain utility (EUU) theory describes a decision-maker who assigns probabilities to some events, regards others as unquantifiable, and values a prospect by a probability-weighted average of a utility of the best and worst outcomes it may yield on each quantifiable event. I extend EUU to dynamic settings where information arrives before an act is chosen. Outside subjective expected utility, three natural requirements cannot generally coexist: an act's value should not depend on how the resolution of uncertainty is described; choices after a signal should depend only on what remains possible; and optimizing at each realization should yield a plan that is optimal in advance. The recursive extension characterized here secures the last two by associating with every signal, ambiguous or not, a unique system of conditional beliefs, while retaining EUU's treatment of risk and ambiguity. For unambiguous signals it reduces to static EUU; otherwise an act's value depends on how information is revealed. A tractable subclass accommodates choice patterns, such as Machina's reflection example, that many ambiguity models cannot.

1. Introduction

Gul and Pesendorfer (2014) provide behavioral foundations for a (static) model of choice under uncertainty that they dub Expected Uncertain Utility (EUU). When choosing between Savage acts, an EUU maximizer is endowed with a prior μ defined on a σ -algebra $\mathcal{E}$ of ideal events. Gul and Pesendorfer interpret ideal events as those aspects of uncertainty that the decision-maker can quantify precisely with a probability.

To evaluate an arbitrary act f , he first forms *ideal* (that is, μ -measurable) greatest lower and least upper bounds $\underline{f}^\mu$ and $\bar{f}^\mu$ that bound the range of possible outcomes he cannot quantify precisely with his prior.¹ The *expected uncertain utility* of f is then given by

$$\int u(\underline{f}^\mu, \bar{f}^\mu) d\mu, \quad (1)$$

where the *interval utility* $u(x, y)$ is the utility assigned to an unquantifiable uncertain prospect with prizes lying between x and y .

Gul and Pesendorfer show their model can account for many observed deviations from subjective expected utility documented in Ellsberg-style experiments, including those considered by Machina (2009).

1.1. A dynamic setting: information decision problems

Since applications of decision theoretic models in economics frequently involve the sequential resolution of uncertainty, this raises the question: how should a static model like EUU be extended to a dynamic setting? The approach I take is to consider an EUU

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¹ That is, $\underline{f}^\mu$ (respectively, $\bar{f}^\mu$) is the largest (respectively, smallest) measurable function that is smaller (respectively, larger) than f .

maximizer facing what I refer to as an *information decision problem* (IDP). As in Gul and Pesendorfer's (2014) framework, the objects of choice are still (Savage) acts. In addition to specifying a menu of available acts, each IDP includes a signal (structure) σ , modeled as a function from the state space to a space of (conceivable) signal realizations.² As the EUU maximizer's actual choice from the menu is made *after* he observes the realization of the signal, *ex ante* he formulates a *plan (of action)* that assigns to each possible realization of the signal an act from the menu. Each feasible plan may be associated with a *signal-act*, a pair comprising the IDP's signal and the act induced by the plan. The EUU maximizer's (ex ante) optimal feasible plan can then be determined from his preference relation $\succeq$ over the set of signal-acts induced by feasible plans. An IDP with n possible signal realizations and m available acts has m^n feasible plans. If each plan induces a distinct act, determining an optimal plan directly from $\succeq$ requires $m^n - 1$ preference comparisons.

The following example makes the distinction between feasible plans and induced (reduced-form) acts explicit, and also illustrates the combinatorial growth in the number of feasible plans as the signal is refined.

Example 1. Let $\Omega = [0, 1]$ and $X = \{0, 1\}$. Let the signal σ have two realizations s', s'' with information cells

$$\sigma^{-1}(s') = [0, \tfrac{1}{2}), \quad \sigma^{-1}(s'') = [\tfrac{1}{2}, 1].$$

Consider the menu $M = \{f, g, h\}$ where

$$f(\omega) = \mathbf{1}\{\omega < \tfrac{1}{2}\}, \quad g(\omega) \equiv 0, \quad h(\omega) = \mathbf{1}\{\omega \geq \tfrac{1}{2}\}.$$

A *plan* assigns to each realization of σ an act from M . Hence there are $3^2 = 9$ feasible plans. Each plan induces a reduced-form act obtained by applying the prescribed act on the realized information cell. For instance, the plan “choose f if s' , else choose g ” induces the act

$$\omega \mapsto \begin{cases} f(\omega), & \omega \in [0, \tfrac{1}{2}), \\ g(\omega), & \omega \in [\tfrac{1}{2}, 1], \end{cases}$$

which coincides with f (since $f(\omega) = 0$ on $[\tfrac{1}{2}, 1]$). Likewise, the plan “always choose f ” also induces f . Thus distinct plans can induce the same reduced-form act, even though the agent chooses contingently on the signal. By contrast, the plan “choose f if s' , else choose h ” induces the constant act $\mathbf{1}$.

More generally, if a signal σ has n realizations and there are m alternatives in a menu M , then there are m^n feasible plans in the IDP (σ, M) . This combinatorial growth motivates using signal-realization contingent (*interim*) preferences to compute optimal plans recursively rather than comparing all feasible plans directly.

1.2. Three desiderata, and where dynamic non-EU models must “pay”

The goal of this paper is to provide a dynamic extension of EUU that preserves three desiderata that are natural in a sequential setting. First, the interim evaluation used after observing a signal realization should be *model consistent* with EUU: conditional beliefs should be represented by a (suitably interpreted) conditional version of the prior, and risk and uncertainty preferences should continue to be encoded by the same interval utility u . Second, interim preferences should be *consequentialist*: conditional rankings given a realized information cell should depend only on payoffs on that cell (and not on what would have been chosen in counterfactual information cells).³ Third, ex ante and interim evaluations should be *dynamically consistent*: the ex ante evaluation of a contingent plan should be recursive in its interim evaluations, so that a plan that is interim optimal after every realized signal is optimal ex ante.

Because terms like *recursive*, *consequentialist*, and *dynamically consistent* are used differently across the literature, it is helpful to separate these desiderata from a further, distinct EU-benchmark property: *tree-independence*. To that end, I distinguish three properties relative to expected utility with Bayesian updating:

1. **Tree-independence (ex ante).** Ex ante rankings depend only on the induced (reduced-form) act, not on the signal structure used to describe the resolution of uncertainty.
2. **Within-event consequentialism (interim).** Conditional rankings at a realized information cell depend only on the restriction of acts to that cell, not on payoffs outside it.
3. **Dynamic consistency.** Ex ante evaluation is recursive in the interim evaluations: replacing an act on each reached information cell by its interim certainty equivalent leaves the ex ante ranking of signal-acts unchanged, so a plan that is interim-undominated at every reached cell is optimal ex ante.

Once one departs from subjective expected utility, these three properties cannot in general all be retained. REUU preserves within-event consequentialism and dynamic consistency in this recursive sense, while tree-independence fails outside the ideal-signal case. The stronger converse, that every ex ante optimal plan is interim-undominated at every reached cell, holds for ideal signals but not in general, since the outer evaluation of a non-ideal signal-act can be insensitive to a strict interim improvement confined to one cell. For ideal signals the tension disappears altogether: tree-independence is restored, and the dynamic model collapses to static EUU.

² I assume each menu is finite as is the range of each signal.

³ Terminology varies: some authors use “consequentialism” to refer to a property of the decision tree or to a stronger invariance with respect to the description of the information structure. Throughout, I use “consequentialist” in the within-event sense spelled out here.

1.3. Ideal signals

Suppose the signal σ is ideal. Then for each possible realization s , the event $\sigma^{-1}(s)$ is in $\mathcal{E}$; that is, the associated information cell is ideal. Since the EUU maximizer does not perceive there to be any ambiguity with regard to the resolution of uncertainty of an ideal signal, it is natural to require that the restriction of $\succsim$ to signal-acts involving ideal signals *agrees* with his static preferences. Thus, applying the representation of static preferences from (1), for any pair of ideal signals σ and $\hat{\sigma}$ and any pair of acts f and $\hat{f}$,

$$(\sigma, f) \succsim (\hat{\sigma}, \hat{f}) \iff \int u(\underline{f}^\mu, \bar{f}^\mu) d\mu \geq \int u(\underline{\hat{f}}^\mu, \bar{\hat{f}}^\mu) d\mu. \quad (2)$$

Hence, the *indirect* utility the EUU maximizer assigns to an IDP with an ideal signal—with which he can compare the desirability of this IDP with other IDPs with ideal signals—is simply the static expected uncertain utility of the act induced by an optimal plan.

In addition, an optimal plan can be constructed *recursively*. For each realization s of σ with $\mu(\sigma^{-1}(s)) > 0$, let $\succsim_s^\sigma$ denote an (interim) preference relation over acts derived from $\succsim$ by setting

$$f \succsim_s^\sigma \hat{f} \text{ if } (\sigma, f) \succsim (\sigma, \hat{f}_{\sigma^{-1}(s)}), \quad (3)$$

where $\hat{f}_{\sigma^{-1}(s)}f$ denotes the act that agrees with $\hat{f}$ on the event $\sigma^{-1}(s)$ and with f elsewhere. Straightforward algebraic manipulation of (1) yields the representation

$$V_s^\sigma(f) = \int_{\sigma^{-1}(s)} u(\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma}) d\mu_s^\sigma, \quad (4)$$

where μ_s^σ is the posterior belief over the (relative) sigma-algebra $\mathcal{E}_s^\sigma := \{E \cap \sigma^{-1}(s) : E \in \mathcal{E}\}$, derived from μ by setting

$$\mu_s^\sigma(E \cap \sigma^{-1}(s)) := \frac{\mu(E \cap \sigma^{-1}(s))}{\mu(\sigma^{-1}(s))},$$

and where $f^{\mu_s^\sigma}$ (respectively, $\bar{f}^{\mu_s^\sigma}$) is the largest (respectively, smallest) function measurable with respect to μ_s^σ that is smaller (respectively, larger) than f on the information cell $\sigma^{-1}(s)$ and agrees with f outside $\sigma^{-1}(s)$.⁴ Given these signal-realization contingent (interim) preferences, a plan is optimal if and only if for each non-null realization the act assigned by the plan for that realization is, according to these interim preferences, undominated by every other available act in the menu M ; choices on null information cells are behaviorally irrelevant. Hence if the signal has n non-null realizations and the menu has m available acts then finding an optimal plan requires only (at most) $n \times (m-1)$ (interim) preference comparisons.⁵

Summary (ideal signals). In the benchmark case of ideal signals, the induced interim preferences $\succsim_s^\sigma$ are EUU preferences with Bayesian posteriors and the same interval utility u . Consequently, they satisfy the desiderata of model consistency, within-event consequentialism, and dynamic consistency. Moreover, because no non-ideal residual arises when every information cell $\sigma^{-1}(s)$ is ideal, ex ante evaluation in this benchmark is tree-independent and coincides with the static EUU functional.

More explicitly, if we let v denote the Bernoulli utility derived from the interval utility by setting $v(x) := u(x, x)$, then it follows from (4) that $f \sim_s^\sigma v^{-1} \circ V_s^\sigma(f)$. That is, the outcome $v^{-1} \circ V_s^\sigma(f)$ is the *conditional certainty equivalent* of the act f given s is the realization of the signal σ . Now define $c_s^\sigma(f) := v^{-1}(V_s^\sigma(f))$ for each non-null realization and construct an act f_σ adapted to σ by setting its restriction on each information cell with positive μ -probability equal to $c_s^\sigma(f)$ and its value on any null information cell equal, arbitrarily, to ℓ . In the ideal-signal benchmark, this yields the following convenient decomposition of the ex ante value:

$$\begin{aligned} V(\sigma, f) &= \int u(\underline{f}^\mu, \bar{f}^\mu) d\mu = \sum_{s: \mu(\sigma^{-1}(s)) > 0} V_s^\sigma(f) \mu(\sigma^{-1}(s)) \\ &= \sum_{s: \mu(\sigma^{-1}(s)) > 0} v(c_s^\sigma(f)) \mu(\sigma^{-1}(s)) = \int v \circ f_\sigma d\mu. \end{aligned}$$

1.4. Extension to arbitrary signals

The model developed in Section 2 is one in which this recursive evaluation can be adapted to work for *all* IDPs. In particular, I retain the certainty-equivalent construction $c_s^\sigma(f) = v^{-1}(V_s^\sigma(f))$ used above to build the σ -adapted act f_σ . For an arbitrary signal, the interim preference $\succsim_s^\sigma$ ranks acts by the sure outcome that can replace their restriction to the information cell in every continuation (Axiom 2 in Section 3); when $\sigma^{-1}(s)$ is ideal this agrees with (3). The novelty is that when $\sigma^{-1}(s) \notin \mathcal{E}$ the posterior belief μ_s^σ is not obtained by the standard Bayesian update, but instead is generated by a modified updating rule that extends Bayesian conditioning to non-ideal information cells. Formally, for each signal σ and each of its non-null realizations s (a notion made precise in Section 3), the

⁴ Notice that, since $\sigma^{-1}(s) \in \mathcal{E}$, $f^{\mu_s^\sigma}$ (respectively, $\bar{f}^{\mu_s^\sigma}$) agrees with f^μ (respectively, $\bar{f}^\mu$) on the information cell $\sigma^{-1}(s)$. However, this will turn out not to be the case in the sequel when I extend the analysis to signals with information cells that need not be ideal.

⁵ The number of comparisons could be fewer should one or more of the acts agree with each other on one or more of the signal's information cells.

EUU maximizer attaches a posterior belief μ_s^σ over the sigma-algebra $\mathcal{E}_s^\sigma = \{E \cap \sigma^{-1}(s) : E \in \mathcal{E}\}$ that, in conjunction with the interval utility u from (1), characterizes an EUU representation of the form (4) for the interim preferences $\succsim_s^\sigma$.

Using these signal-realization contingent functionals the EUU maximizer can compute an optimal plan by assigning to each realization an act from the menu that maximizes the corresponding conditional expected uncertain utility. Apart from the modification to the updating rule needed to handle non-ideal information cells, these interim preferences continue to satisfy model consistency, (within-event) consequentialism, and dynamic consistency in the recursive sense above. Furthermore, I show that the utility the EUU maximizer assigns to an arbitrary signal-act (σ, f) may be expressed as the expected *uncertain* utility of an act f_σ adapted to σ and defined by setting its restriction on each information cell $\sigma^{-1}(s)$ equal to $v^{-1} \circ V_s^\sigma(f)$. That is,

$$V(\sigma, f) = \int u\left(\underline{f_\sigma}^\mu, \bar{f_\sigma}^\mu\right) d\mu. \quad (5)$$

Correspondingly, if we consider an arbitrary IDP with signal σ and menu M for which the feasible plan $\phi : \{s : \sigma^{-1}(s) \text{ is non-null}\} \rightarrow M$ is optimal, then the *indirect* utility assigned by the EUU maximizer to this IDP, with which he can compare its desirability with respect to *any* other IDP, may be expressed as the expected *uncertain* utility of an act ϕ_σ adapted to σ that is defined by setting its restriction on each information cell $\sigma^{-1}(s)$ equal to $v^{-1} \circ V_s^\sigma(\phi(s))$.

As I noted above, for ideal signals, ex ante evaluation collapses to static EUU (cf. (2)). Outside this benchmark class, however, the analogous equality need not hold.⁶ I do not view this as a defect, but rather as the natural implication of how ambiguity is perceived in a sequential environment. Channeling the insights of Frisch and Baron (1988), ambiguity about a realized information cell arises from missing information that is relevant and could in principle be known. Once the information cell is known to have obtained, the only remaining ambiguity concerns the probabilities of subevents of that cell; ambiguity that would have been faced under other, now counterfactual, cells is no longer decision relevant at the interim stage. However, *ex ante*, before the realization is learned, that ambiguity has still to be borne, and its impact is precisely what is captured by the dependence of $V(\sigma, f)$ on the signal structure in (5).

1.5. Roadmap

Section 2 formalizes information decision problems and derives the REUU evaluation of signal-acts, first for ideal signals and then for arbitrary signals via the uniform conditional belief system. Section 3 provides the axiomatic foundations linking ex ante and interim evaluations and yielding the representation theorem. Section 4 studies the neoadditive subclass, giving a parsimonious “neo-EU” representation. Section 5 situates the approach relative to existing decision-theoretic and statistical updating literatures.

2. The model

Adopting Gul and Pesendorfer’s setting of purely subjective uncertainty, Ω (with generic element ω) denotes the state space, subsets of which are referred to as events. The non-degenerate interval $X = [\ell, m]$ comprises the set of (monetary) outcomes and $\mathcal{F}$, the set of mappings from Ω to X , is the set of (Savage) acts, the objects of choice. Each outcome $x \in X$ is also identified with the (constant) act f in which $f(\omega) = x$ for all ω . And with further (albeit fairly standard) abuse of notation, X will also refer to the set of constant acts.

For any pair of events $B, E \subseteq \Omega$, $B \setminus E$ shall denote the set of elements that are in B but not in E . For any pair of acts f and $\hat{f}$ in $\mathcal{F}$ and any event $B \subseteq \Omega$, $f_B \hat{f}$ is identified with the act that agrees with f on B and with $\hat{f}$ on $\Omega \setminus B$.

Let S denote the universe of conceivable realizations for any possible signal. It includes the *null* realization o .

The first component of an *information decision problem* (IDP) is a signal (structure) described by a function $\sigma : \Omega \rightarrow S$ with finite range, with the interpretation that the decision-maker learns the realization of the signal is s should the event $\sigma^{-1}(s)$ obtain. Let Σ denote the set of signals which I take to be a non-empty subset of functions from Ω to S . In particular, I assume Σ contains a *null* (that is, *uninformative*) signal, denoted $\mathbf{0}$, a constant mapping in which $\mathbf{0}(\omega) = o (\in S)$ for all ω in Ω . The second component of an IDP is a finite menu of acts $M \subset \mathcal{F}$, $|M| < \infty$. Let $\mathcal{M}$ denote the set of menus. As any signal $\sigma \in \Sigma$ may be paired with any menu $M \in \mathcal{M}$ to form the IDP (σ, M) , the set of IDPs thus comprises the product set $\Sigma \times \mathcal{M}$.

A *plan* (of action) ϕ assigns to each $s \in S$ an act $\phi_s \in \mathcal{F}$. A plan of action ϕ is *feasible* in the IDP (σ, M) , if $\phi_s \in M$ for each $s \in \sigma(\Omega)$. With a slight abuse of the notation, each feasible plan in an IDP is *identified* with the act it generates. That is, a feasible plan ϕ for the IDP (σ, M) is identified with the act defined by setting $\phi(\omega) := \phi_s(\omega)$ if $\omega \in \sigma^{-1}(s)$.

For each IDP (σ, M) in $\Sigma \times \mathcal{M}$, let $\Phi(\sigma, M) \subset \mathcal{F}$ denote the set of acts generated by feasible plans in that IDP. I refer to any pair (σ, f) in $\Sigma \times \mathcal{F}$ as a *signal-act* and note it may be viewed as the subclass of IDPs in which the menu contains just a single option. Thus I view and treat $\Sigma \times \mathcal{F}$ (the set of signal-acts) as a subset of $\Sigma \times \mathcal{M}$ (the set of IDPs), identifying each signal-act (σ, f) with the IDP $(\sigma, \{f\})$ whose menu is the singleton $\{f\}$.

The decision-maker is characterized by a binary relation $\succsim$ defined over $\Sigma \times \mathcal{F}$. For each particular signal σ in Σ , I refer to the restriction of $\succsim$ to $\{\sigma\} \times \mathcal{F}$ as his *ex ante preferences* with respect to that signal. Finally, I refer to any signal-act in $\{\mathbf{0}\} \times \mathcal{F}$ as *static* and the restriction of $\succsim$ to $\{\mathbf{0}\} \times \mathcal{F}$ as the decision-maker’s *static preferences*.

⁶ Indeed, the equality only holds for a signal-act involving a signal that is not adapted to $\mathcal{E}$ when the act is constant on every information cell associated with the signal that is not ideal.

A *prior* is a countably-additive, complete and non-atomic probability measure defined on a σ -algebra of subsets of Ω . Let Π denote the set of all priors with generic element π . For each π in Π (with domain $\mathcal{E}^\pi$) let $\mathcal{F}^\pi$ denote the set of acts that are measurable with respect to π (that is, adapted to $\mathcal{E}^\pi$).

Like his counterpart in Gul and Pesendorfer (2014), our decision-maker's static preferences are characterized by a pair $\langle \mu, u \rangle$, where:

- (i) $\mu \in \Pi$ with domain $\mathcal{E}$ is the decision-maker's prior;⁷ and,
- (ii) $u: \{(x, y) \in X \times X: x \leq y\} \rightarrow \mathbb{R}$ is his interval utility, a continuous function for which $u(x, y) > u(x', y')$ whenever $x > x'$ and $y > y'$.

Adopting Gul and Pesendorfer's (2014) nomenclature, I will refer to any event E (respectively, any act g) in $\mathcal{E}$ (respectively, in $\mathcal{F}^\mu$) as *ideal*.

For any ideal act $g \in \mathcal{F}^\mu$ its static subjective expected utility is given by $\int v(g) d\mu$, where $v: X \rightarrow \mathbb{R}$ is the (increasing and continuous) Bernoulli utility defined by setting $v(x) := u(x, x)$. For any arbitrary act f , however, although it may not be measurable with respect to μ , the decision-maker can always compute ideal greatest-lower- and least-upper-bounds, that together form a (μ -)measurable *envelope* for that act.

Definition 1 (Ideal envelope). Fix a prior $\mu \in \Pi$. For each act $f \in \mathcal{F}$, the pair of acts $\underline{f}^\mu, \bar{f}^\mu \in \mathcal{F}^\mu$ constitutes an *ideal envelope* if $\underline{f}^\mu \leq f \leq \bar{f}^\mu$ and for any $\hat{f} \in \mathcal{F}^\mu$:

- (i) if $f \geq \hat{f}$, then

$$\mu(\{\omega \in \Omega: \underline{f}^\mu(\omega) < \hat{f}(\omega)\}) = 0;$$

- (ii) if $f \leq \hat{f}$, then

$$\mu(\{\omega \in \Omega: \bar{f}^\mu(\omega) > \hat{f}(\omega)\}) = 0.$$

Lemma 1 in Gul and Pesendorfer (2014, p5) establishes that for each act f there exists an essentially unique ideal envelope. The utility the decision-maker assigns the signal-act $(\mathbf{0}, f)$ is the (static) expected uncertain utility of f . That is,

$$V(\mathbf{0}, f) = \int u(\underline{f}^\mu, \bar{f}^\mu) d\mu, \quad (6)$$

represents the restriction of $\succsim$ to $\{\mathbf{0}\} \times \mathcal{F}$ (his static preferences). Henceforth, I refer to such a decision-maker as an EUU maximizer.

Turning now to arbitrary signal-acts, recall from the introduction, I proposed that for each signal σ and each non-null realization s of σ , the EUU maximizer attaches a posterior belief μ_s^σ defined over the events in the (relative) sigma-algebra

$$\mathcal{E}_s^\sigma = \{E \cap \sigma^{-1}(s): E \in \mathcal{E}\}. \quad (7)$$

For each posterior μ_s^σ , let $\mathcal{F}^{\mu_s^\sigma} \subset \mathcal{F}$ denote the set of acts whose restriction to the information cell $\sigma^{-1}(s)$ is measurable with respect to μ_s^σ . Analogous to Definition 1 above of an ideal envelope, the next definition specifies a μ_s^σ -measurable envelope for the restriction of an act to the information cell $\sigma^{-1}(s)$.

Definition 2 (Conditional measurable envelope). Fix a prior $\mu \in \Pi$, a signal $\sigma \in \Sigma$, a realization $s \in \sigma(\Omega)$, and a probability measure μ_s^σ on $\mathcal{E}_s^\sigma$ (in what follows, the posterior of Definition 4, which is defined whenever $p_s^\sigma > 0$). For each act $f \in \mathcal{F}$, the pair of acts $\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma} \in \mathcal{F}^{\mu_s^\sigma}$ that agree with f on $\Omega \setminus \sigma^{-1}(s)$ constitutes a μ_s^σ -measurable envelope if $\underline{f}^{\mu_s^\sigma} \leq f \leq \bar{f}^{\mu_s^\sigma}$ and for any $\hat{f} \in \mathcal{F}^{\mu_s^\sigma}$ that agrees with f on $\Omega \setminus \sigma^{-1}(s)$:

- (i) if $f \geq \hat{f}$, then

$$\mu_s^\sigma(\{\omega \in \sigma^{-1}(s): \underline{f}^{\mu_s^\sigma}(\omega) < \hat{f}(\omega)\}) = 0;$$

- (ii) if $f \leq \hat{f}$, then

$$\mu_s^\sigma(\{\omega \in \sigma^{-1}(s): \bar{f}^{\mu_s^\sigma}(\omega) > \hat{f}(\omega)\}) = 0.$$

To describe how I propose the decision-maker forms these posterior beliefs entails the use of the *ideal split* generated by the preimage of a signal. The specification of this partition of the state space requires some additional definitions and notation, starting with the inner-measure of μ , denoted by μ_* and defined by setting

$$\mu_*(B) := \sup_{E \subseteq B: E \in \mathcal{E}} \mu(E) \text{ for each } B \subseteq \Omega.$$

Since μ is countably additive, the supremum is attained. I shall refer to an ideal event $[B]_*$ in $\mathcal{E}$ as an *inner-sleeve* of B if $[B]_* \subseteq B$ and $\mu([B]_*) = \mu_*(B)$. An inner-sleeve is unique only up to μ -null events and may be viewed as a largest ideal subset of B up to such events. Throughout, equalities and inclusions involving inner-sleeves and elements of an ideal split are understood up to μ -null events whenever the representatives are not uniquely determined; representatives may always be chosen so that the elements of an ideal split are pairwise disjoint and exhaust Ω .

⁷ To ease the notational clutter, I let $\mathcal{E}$ rather than $\mathcal{E}^\mu$ denote the domain of the decision-maker's prior.

Definition 3 (A Signal's Ideal Split). Fix a prior $\mu \in \Pi$. For each signal $\sigma \in \Sigma$, after choosing representatives as above, its *ideal split*, denoted by the collection of ideal events $\{E_Q^\sigma \in \mathcal{E} : Q \subseteq \sigma(\Omega), Q \neq \emptyset\}$, is the partition of the state space inductively defined as follows:

1. For each realization $s \in \sigma(\Omega)$, set $E_{\{s\}}^\sigma := [\sigma^{-1}(s)]_*$.
2. For each $Q \subseteq \sigma(\Omega)$ such that $|Q| > 1$, set

$$E_Q^\sigma := [\sigma^{-1}(Q)]_* \setminus \left(\bigcup_{Q' \subset Q} E_{Q'}^\sigma \right).$$

I refer to E_Q^σ as the σ -marginal inner-sleeve of the set of signal realizations Q .

As an illustration, consider the ideal split generated by the preimage of a binary signal $\hat{\sigma}$ that maps each state to one of two possible realizations, s' or s'' . The ideal split is a three element partition of the state-space

$$\left\{ \begin{array}{c} E_{\{s'\}}^{\hat{\sigma}} \\ \parallel \\ [\hat{\sigma}^{-1}(s')]_* \end{array} , \begin{array}{c} E_{\{s''\}}^{\hat{\sigma}} \\ \parallel \\ [\hat{\sigma}^{-1}(s'')]_* \end{array} , \begin{array}{c} E_{\{s', s''\}}^{\hat{\sigma}} \\ \parallel \\ \Omega \setminus ([\hat{\sigma}^{-1}(s')]_* \cup [\hat{\sigma}^{-1}(s'')]_*) \end{array} \right\}.$$

The first (respectively, second) element corresponds to a largest ideal subset, up to null events, in which the signal's realization is s' (respectively, s''). The third element corresponds to the $\hat{\sigma}$ -marginal inner-sleeve of the set of realizations $\hat{\sigma}(\Omega)$ ($= \{s', s''\}$). All the EUU maximizer can discern from this element of the ideal split is that the realization will be either s' or s'' . However, he is unable to attribute any fraction of the probability his prior assigns to this element of the split to either s' or s'' alone obtaining.

More generally for an arbitrary signal σ and for each non-empty set of its realizations $Q \subseteq \sigma(\Omega)$, recalling the approach of [Dempster \(1967\)](#) and [Shafer \(1976\)](#), I interpret $\mu(E_Q^\sigma)$ (the probability assigned by the prior to the σ -marginal inner-sleeve of Q) as measuring the weight the decision-maker places on the evidence that directly supports the realization of the signal residing in Q that *cannot be further refined ex ante in terms of any of its strict subsets*.

Since he anticipates after learning the realization of the signal he will then be assigning the information cell associated with that realization a (precise) posterior belief of 1 and all the other information cells (precise) posterior beliefs of 0, he treats the partition of information cells as *conditionally ideal*. Furthermore, given the particular realization s , he treats as *conditionally ideal* the intersection of each (unconditionally) ideal event with the information cell $\sigma^{-1}(s)$. That is, for each realization s of the signal, he treats as conditionally ideal all the events in the (relative) sigma-algebra $\mathcal{E}_s^\sigma$.

To help motivate the updating rule that appears in [Definition 4](#) below, first notice for each non-empty set of the signal realizations $Q \subset \sigma(\Omega)$ and any pair of realizations s' and s'' that both reside in Q , we have

$$\left(\mathbf{0}, (x_{\sigma^{-1}(s')}y)_{E \cap E_Q^\sigma} f \right) \sim \left(\mathbf{0}, (x_{\sigma^{-1}(s'')}y)_{E \cap E_Q^\sigma} f \right),$$

for all ideal events $E \in \mathcal{E}$, all pairs of outcomes x and y , and all acts f . To see this, notice the application of the representation of the static preferences in equation (6) yields

$$\begin{aligned} & V\left(\mathbf{0}, (x_{\sigma^{-1}(s')}y)_{E \cap E_Q^\sigma} f\right) - V\left(\mathbf{0}, (x_{\sigma^{-1}(s'')}y)_{E \cap E_Q^\sigma} f\right) \\ &= \mu(E \cap E_Q^\sigma)u(\min\{x, y\}, \max\{x, y\}) + C_f \\ & \quad - \left[\mu(E \cap E_Q^\sigma)u(\min\{x, y\}, \max\{x, y\}) + C_f \right] = 0, \end{aligned}$$

where C_f denotes the common contribution to the two values of the integrand $u(\bar{f}^\mu, \bar{f}^\mu)$ over $\Omega \setminus (E \cap E_Q^\sigma)$, on which the two composite acts coincide, and where the first term uses that, for each $s \in Q$ with $|Q| \geq 2$, the envelope pair of $x_{\sigma^{-1}(s)}y$ on $E \cap E_Q^\sigma$ is $(\min\{x, y\}, \max\{x, y\})$ almost everywhere, since neither $\sigma^{-1}(s)$ nor its complement contains a non-null ideal subset of E_Q^σ ([Lemma 2\(d\)](#) in [A](#)). This means from his static preferences we see the EUU maximizer is indifferent between betting on (respectively, against) any event in the collection $\{E \cap E_Q^\sigma \cap \sigma^{-1}(s) : s \in Q\}$ versus betting on (respectively, against) any other event in the collection. In other words, all these events that together form a $|Q|$ -element partition of the ideal event $E \cap E_Q^\sigma$ are *exchangeable* with each other. So although these events need not be measurable with respect to the prior μ , the EUU maximizer's conditional belief system embodies a "principle of indifference" property by allocating the probability $\mu(E \cap E_Q^\sigma)$ *uniformly* across its intersections with the information cells associated with the realizations in Q .

The logic of this reasoning underpins the next definition of the *uniform conditional belief system* that the EUU maximizer associates with a signal. It comprises a pseudo-prior (that renders all the information cells of the signal unambiguous) and in addition, for each realization for which the pseudo-prior is strictly positive, it specifies a conditional probability (his *posterior*) defined over those events he views as conditionally ideal.

Definition 4 (Uniform Conditional Belief System). Fix a prior $\mu \in \Pi$. The *uniform conditional belief system* associated with the signal $\sigma \in \Sigma$ comprises for each realization $s \in \sigma(\Omega)$:

(i) a pseudo-prior $p_s^\sigma \in [0, 1]$ given by

$$p_s^\sigma := \sum_{Q \subseteq \sigma(\Omega): s \in Q} \frac{\mu(E_Q^\sigma)}{|Q|};$$

and, in addition, whenever $p_s^\sigma > 0$,

(ii) a posterior μ_s^σ defined on $\mathcal{E}_s^\sigma = \{E \cap \sigma^{-1}(s) : E \in \mathcal{E}\}$ and obtained by setting for each event $E \in \mathcal{E}$:

$$\mu_s^\sigma(E \cap \sigma^{-1}(s)) := \frac{\sum_{Q \subseteq \sigma(\Omega): s \in Q} \mu(E \cap E_Q^\sigma) / |Q|}{p_s^\sigma}.$$

Remark 1. Notice that $\mu_s^\sigma(E \cap \sigma^{-1}(s))$ is well-defined since for any pair of ideal events E and $\hat{E}$ in $\mathcal{E}$, if $E \cap \sigma^{-1}(s) = \hat{E} \cap \sigma^{-1}(s)$ then it readily follows that $\mu(E \cap E_Q^\sigma) = \mu(\hat{E} \cap E_Q^\sigma)$ for all $Q \subseteq \sigma(\Omega)$ such that $s \in Q$.

Remark 2. Fix a signal $\sigma \in \Sigma$. If for any of its realizations $s \in \sigma(\Omega)$ the information set $\sigma^{-1}(s)$ is ideal, it may itself be chosen as its inner-sleeve $[\sigma^{-1}(s)]_*$. It follows that for every $Q \subseteq \sigma(\Omega)$ with $s \in Q$ and $|Q| > 1$, the corresponding element E_Q^σ of the ideal split is null and may be represented by the empty set. Hence

$$p_s^\sigma = \mu(\sigma^{-1}(s)).$$

If, in addition, $p_s^\sigma > 0$ (equivalently, the realization is non-null), then

$$\mu_s^\sigma(E \cap \sigma^{-1}(s)) = \frac{\mu(E \cap \sigma^{-1}(s))}{\mu(\sigma^{-1}(s))}.$$

Thus, for an ideal non-null realization, the posterior corresponds to the standard Bayesian update of the prior conditional on that realization obtaining.

Remark 3. The uniform conditional belief system exhibits the following “law of total probability”. For each $E \in \mathcal{E}$:

$$\sum_{s \in \sigma(\Omega): p_s^\sigma > 0} p_s^\sigma \mu_s^\sigma(E \cap \sigma^{-1}(s)) = \mu(E).$$

To see this, applying the definitions above we have

$$\begin{aligned} \sum_{s \in \sigma(\Omega): p_s^\sigma > 0} p_s^\sigma \mu_s^\sigma(E \cap \sigma^{-1}(s)) &= \sum_{s \in \sigma(\Omega): p_s^\sigma > 0} \left(\sum_{Q \subseteq \sigma(\Omega): s \in Q} \frac{\mu(E \cap E_Q^\sigma)}{|Q|} \right) \\ &= \sum_{Q \subseteq \sigma(\Omega)} \left(\sum_{s \in Q} \frac{\mu(E \cap E_Q^\sigma)}{|Q|} \right) = \sum_{Q \subseteq \sigma(\Omega)} \mu(E \cap E_Q^\sigma) = \mu(E). \end{aligned}$$

Furthermore, substituting Ω for E yields $\sum_s p_s^\sigma \underbrace{\mu_s^\sigma(\sigma^{-1}(s))}_{=1} = \mu(\Omega) = 1$.

Example 2 (A non-ideal binary signal). Set $\Omega := [0, 1] \times [0, 1]$ and let $\mathcal{B}$ denote the collection of Borel subsets of $[0, 1]$. Let

$$\mathcal{E}_0 := \{[0, 1] \times B : B \in \mathcal{B}\}, \quad \mu_0([0, 1] \times B) := \lambda(B),$$

where λ is Lebesgue measure on $[0, 1]$, and take $(\mathcal{E}, \mu)$ to be the completion of $(\mathcal{E}_0, \mu_0)$, so that $\mu \in \Pi$. Up to μ -null events, ideal events are full-length horizontal bands.

Fix numbers $0 < a < b < 1$ and $\alpha \in (0, 1)$. Define a binary signal σ with realizations $\{s', s''\}$ by specifying the information cell for s' as

$$\sigma^{-1}(s') = A := ([0, 1] \times [b, 1]) \cup ([0, \alpha] \times [a, b]),$$

and hence $\sigma^{-1}(s'') = \Omega \setminus A$.⁸

The ideal (measurable) split. An inner-sleeve of A (unique up to null events) is

$$[A]_* = [0, 1] \times [b, 1],$$

and an inner-sleeve of $\Omega \setminus A$ is

$$[\Omega \setminus A]_* = [0, 1] \times [0, a].$$

⁸ Geometrically: $\sigma^{-1}(s')$ is the top horizontal band $[0, 1] \times [b, 1]$ together with the left block of the middle band $[0, 1] \times [a, b]$; its complement is the bottom band $[0, 1] \times [0, a]$ together with the right block of the middle band.

Accordingly, the induced ideal split has three (ideal) cells:

$$E_{\{s'\}}^\sigma = [0, 1] \times [b, 1], \quad E_{\{s''\}}^\sigma = [0, 1] \times [0, a], \quad E_{\{s', s''\}}^\sigma = [0, 1] \times [a, b].$$

The third cell $E_{\{s', s''\}}^\sigma$ is the “residual” region: ex ante it can only be assigned to the set $\{s', s''\}$ (it is not ideal-refinable into singletons), even though ex post the signal realization determines whether the left or right block of it is reached.

Pseudo-priors (uniform allocation over the residual set). Definition 4 assigns pseudo-priors

$$p_{s'}^\sigma = \mu(E_{\{s'\}}^\sigma) + \frac{1}{2} \mu(E_{\{s', s''\}}^\sigma), \quad p_{s''}^\sigma = \mu(E_{\{s''\}}^\sigma) + \frac{1}{2} \mu(E_{\{s', s''\}}^\sigma).$$

Since $\mu([0, 1] \times [c, d]) = d - c$, we have

$$\mu(E_{\{s'\}}^\sigma) = 1 - b, \quad \mu(E_{\{s''\}}^\sigma) = a, \quad \mu(E_{\{s', s''\}}^\sigma) = b - a,$$

and therefore

$$p_{s'}^\sigma = (1 - b) + \frac{1}{2}(b - a), \quad p_{s''}^\sigma = a + \frac{1}{2}(b - a),$$

with $p_{s'}^\sigma + p_{s''}^\sigma = 1$.

Posteriors on conditionally ideal events. Upon observing s' , the posterior $\mu_{s'}^\sigma$ on $\mathcal{E}_{s'}^\sigma := \{E \cap A : E \in \mathcal{E}\}$ is defined by

$$\mu_{s'}^\sigma(E \cap A) = \frac{\mu(E \cap E_{\{s'\}}^\sigma) + \frac{1}{2} \mu(E \cap E_{\{s', s''\}}^\sigma)}{p_{s'}^\sigma}, \quad E \in \mathcal{E},$$

and similarly, upon observing s'' ,

$$\mu_{s''}^\sigma(E \cap (\Omega \setminus A)) = \frac{\mu(E \cap E_{\{s''\}}^\sigma) + \frac{1}{2} \mu(E \cap E_{\{s', s''\}}^\sigma)}{p_{s''}^\sigma}, \quad E \in \mathcal{E}.$$

To see “where this bites” take the ideal test event $E_t = [0, 1] \times [0, t]$ with $t \in [a, b]$. Then $E_t \cap E_{\{s'\}}^\sigma = \emptyset$ and $E_t \cap E_{\{s', s''\}}^\sigma = [0, 1] \times [a, t]$, so

$$\mu_{s'}^\sigma(E_t \cap A) = \frac{\frac{1}{2} \mu([0, 1] \times [a, t])}{(1 - b) + \frac{1}{2}(b - a)} = \frac{\frac{1}{2}(t - a)}{(1 - b) + \frac{1}{2}(b - a)}.$$

This makes explicit that probability mass from the residual (middle) band $[0, 1] \times [a, b]$ is shared uniformly across $\{s', s''\}$, while singleton-refinable mass from either the top or bottom band is treated in the usual Bayesian way.

Relation to Dempster–Shafer style optimistic/pessimistic rules. In Dempster–Shafer updating, the residual mass attached to a set of realizations Q can be converted into lower/upper bounds (belief/plausibility) or, equivalently, allocated in an optimistic or pessimistic manner when forming point probabilities. REUU instead selects a uniform split across Q by an invariance/symmetry requirement: within any set of realizations Q that is not further ideal-refinable ex ante, the singletons in Q are treated as exchangeable, so no singleton is privileged ex ante. Comparative statics against “optimistic” or “pessimistic” allocations can be obtained by replacing the uniform split $\mu(E_Q^\sigma)/|Q|$ with an alternative allocation rule over Q ; Example 2 can then be rerun verbatim to show exactly where such alternatives bite.

All the elements required to define the class of Recursive EUU preferences have now been assembled.

Definition 5 (Recursive EUU Preferences). A binary relation $\succsim$ over signal-acts is a *Recursive Expected Uncertain Utility Preference* if there exists an interval utility u and a prior μ (with which we can associate for each signal $\sigma \in \Sigma$ the uniform conditional belief system $\langle (p_s^\sigma)_{s \in \sigma(\Omega)}, (\mu_s^\sigma)_{s : p_s^\sigma > 0} \rangle$), such that $\succsim$ admits a representation of the form:

$$V(\sigma, f) = \int u(\underline{f}_\sigma^\mu, \bar{f}_\sigma^\mu) d\mu,$$

where f_σ is an act adapted to σ that is defined by setting

$$f_\sigma(\omega) := \begin{cases} v^{-1} \circ V_s^\sigma(f) & \text{if } \omega \in \sigma^{-1}(s) \text{ and } p_s^\sigma > 0 \\ \ell & \text{otherwise} \end{cases};$$

where v is the Bernoulli utility derived from u , defined by setting $v(x) := u(x, x)$, and for each realization s such that $p_s^\sigma > 0$, $V_s^\sigma : \mathcal{F} \rightarrow \mathbb{R}$ is defined by setting

$$V_s^\sigma(f) := \int_{\sigma^{-1}(s)} u(\underline{f}_s^\sigma, \bar{f}_s^\sigma) d\mu_s^\sigma.$$

3. Characterization

I begin by assuming the decision-maker's static preferences admit an EUU representation and that ex ante comparisons of signal-acts respect statewise dominance.⁹

Assumption 1 (Static EUU and Monotonicity). The binary relation $\succeq$ is a weak order; that is, it is complete and transitive.

Moreover, the static preferences admit a representation of the form

$$V(\mathbf{0}, f) = \int u(\underline{f}^\mu, \bar{f}^\mu) d\mu$$

for some prior $\mu \in \Pi$ (with domain $\mathcal{E}$) and interval utility u .

Finally, $\succeq$ is *monotone*: for each signal $\sigma \in \Sigma$ and each pair of acts $f, g \in \mathcal{F}$, if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, then $(\sigma, f) \succeq (\sigma, g)$.

Notice that requiring $\succeq$ to be a monotone weak order entails no loss of content in the characterization below, since any binary relation admitting a numerical representation of the form given in Definition 5 is necessarily complete, transitive and monotone.

Recall in the introduction, I noted that Gul and Pesendorfer viewed the set of ideal events (the domain of the prior μ) as ones capturing aspects of the uncertainty that the decision-maker could quantify precisely with a probability. In terms of the static preferences this requires for any ideal event $E \in \mathcal{E}$ that both it and its complement exhibit a form of Savage's property P2: in the terminology of Gul and Pesendorfer (2014), an ideal event is both "left-ideal" and "right-ideal".

Definition 6 (Ideal Event). The event $E \subseteq \Omega$ is *ideal* if for every quartet of acts $f, f', \hat{f}$ and $\tilde{f}$ in $\mathcal{F}$:

$$(\mathbf{0}, f_E \hat{f}) \succeq (\mathbf{0}, f'_E \hat{f}) \implies (\mathbf{0}, f_E \tilde{f}) \succeq (\mathbf{0}, f'_E \tilde{f})$$

and

$$(\mathbf{0}, \hat{f}_E f) \succeq (\mathbf{0}, \hat{f}_E f') \implies (\mathbf{0}, \tilde{f}_E f) \succeq (\mathbf{0}, \tilde{f}_E f').$$

The axioms that link ex ante and interim evaluations are stated with the aid of the following static notions.

An event $B \subset \Omega$ is *null* (with respect to the static preferences) if $(\mathbf{0}, g_B f) \sim (\mathbf{0}, f)$ for all $f, g \in \mathcal{F}$. Let $\mathcal{N}$ denote the set of null events.¹⁰

For each non-null event $B \notin \mathcal{N}$ and each act $f \in \mathcal{F}$, set

$$\underline{x}(f_B) := \sup\{y \in X : \{\omega \in B : f(\omega) < y\} \in \mathcal{N}\}.$$

$$\bar{x}(f_B) := \inf\{z \in X : \{\omega \in B : f(\omega) > z\} \in \mathcal{N}\}.$$

In terms of the EUU maximizer's static preferences, I interpret $\underline{x}(f_B)$ (respectively, $\bar{x}(f_B)$) as the greatest lower (respectively, least upper) bound for the outcomes the EUU maximizer conceives as possibly arising on the restriction of f to the event B . I write $\underline{x}(f)$ for $\underline{x}(f_\Omega)$ and $\bar{x}(f)$ for $\bar{x}(f_\Omega)$.

For each signal $\sigma \in \Sigma$ set

$$\sigma(\Omega)^+ := \{s \in \sigma(\Omega) : \sigma^{-1}(s) \notin \mathcal{N}\}.$$

That is, $\sigma(\Omega)^+$ comprises all of that signal's non-null realizations. Under Assumption 1, a realization is non-null exactly when it receives positive pseudo-prior weight: $\sigma(\Omega)^+ = \{s \in \sigma(\Omega) : p_s^\sigma > 0\}$ (Lemma 2(c) in the Appendix). The interim preferences introduced below and the conditional functionals of Definition 5 are therefore defined on the same set of realizations.

An event D is deemed *diffuse* (with respect to the static preferences) if $D \cap E \neq \emptyset \implies (\Omega \setminus D) \cap E$ for every non-null ideal event E . A diffuse event may be viewed as maximally uncertain since neither it nor its complement contains a non-null ideal subset. Let $\mathcal{D}$ denote the set of diffuse events.

An act h is deemed *diffuse* if for some diffuse event $D \in \mathcal{D}$, $(\mathbf{0}, h) \sim (\mathbf{0}, \bar{x}(h)_D \underline{x}(h))$. Let $\mathcal{H}$ denote the set of diffuse acts. Notice, for any diffuse act $h \in \mathcal{H}$, its static EUU $V(\mathbf{0}, h) = V(\mathbf{0}, \bar{x}(h)_D \underline{x}(h)) = u(\underline{x}(h), \bar{x}(h))$.

In what follows I maintain, in addition to Assumption 1, that at least one diffuse event exists.¹¹

For each signal $\sigma \in \Sigma$ and each non-null realization $s \in \sigma(\Omega)^+$, let $\succeq_s^\sigma$ denote the decision-maker's *interim preference* over acts derived from $\succeq$ by setting

$$f \succeq_s^\sigma g \quad \text{if} \quad (\sigma, f_{\sigma^{-1}(s)} h) \succeq (\sigma, g_{\sigma^{-1}(s)} h) \quad \text{for every } h \in \mathcal{F}.$$

By construction $\succeq_s^\sigma$ is transitive and depends only on how acts behave on the information cell $\sigma^{-1}(s)$; it is complete under Axiom 2 below (Lemma 1), and under the conditions of Theorem 1 it coincides, for ideal signals, with the relation defined in (3) (Remark 4).

To motivate the first axiom, notice that any signal-act in which the act is adapted to the signal has all uncertainty resolved in one stage: once the decision-maker observes the signal realization, the resulting outcome is known, so uncertainty resolves "early." By

⁹ For an axiomatization of static EUU maximization, see Grant et al. (2024), which corrects the original characterization in Gul and Pesendorfer (2014).

¹⁰ An event is null if and only if it is ideal and μ assigns it measure zero; equivalently, its outer measure is zero. See Lemma 2 in A.

¹¹ Gul and Pesendorfer (2014) show that, under the continuum hypothesis, Ω admits a countably infinite partition into diffuse events. No set-theoretic hypothesis is needed in the environment of Example 2, where the vertical strip $[0, \frac{1}{2}] \times [0, 1]$ is diffuse.

contrast, a signal-act involving the uninformative signal $\mathbf{0}$ also resolves all uncertainty in one stage, but only “late.” **Axiom 1** therefore imposes time neutrality: the REUU maximizer is indifferent between these two timing descriptions of the same adapted act. Unlike the compound risk models of [Kreps and Porteus \(1978\)](#), [Epstein and Zin \(1989\)](#) and [Grant et al. \(1998\)](#), REUU therefore rules out an intrinsic strict preference for early rather than late resolution of uncertainty.

Axiom 1 (Time Neutrality). For each signal $\sigma \in \Sigma$: $(\mathbf{0}, f) \sim (\sigma, f)$ for any f in which $\sigma(\omega) = \sigma(\hat{\omega}) \implies f(\omega) = f(\hat{\omega})$.

Next, I require ex ante signal-act preferences to treat each realized information cell in a consequentialist manner: statically irrelevant events remain irrelevant in every signal context, and the restriction of an act to a realized information cell is summarized by a sure outcome that can stand in for it whatever the act does elsewhere.

Axiom 2 (Within-Event Consequentialism). Fix a signal $\sigma \in \Sigma$.

(i) For each realization $s \in \sigma(\Omega)$ and each act $f \in F$ there exists an outcome $c \in X$ such that

$$(\sigma, f_{\sigma^{-1}(s)} y) \sim (\sigma, c_{\sigma^{-1}(s)} y) \quad \text{for every } y \in X,$$

and every such outcome satisfies

$$(\sigma, f_{\sigma^{-1}(s)} h) \sim (\sigma, c_{\sigma^{-1}(s)} h) \quad \text{for every } h \in F.$$

(ii) For each null event $B \in \mathcal{N}$ and all acts $f, g \in F$: $(\sigma, f) \sim (\sigma, g_B f)$.

Lemma 1. Suppose [Assumption 1](#) and [Axioms 1](#) and [2](#) hold. Fix a signal $\sigma \in \Sigma$ and a realization $s \in \sigma(\Omega)^+$. Then:

- (i) for each $f \in F$ there is exactly one outcome, denoted $c_s^\sigma(f)$, such that $(\sigma, f_{\sigma^{-1}(s)} y) \sim (\sigma, c_s^\sigma(f)_{\sigma^{-1}(s)} y)$ for every $y \in X$; thus the conditional certainty equivalent is determined by the constant continuations alone. It depends only on the restriction of f to $\sigma^{-1}(s)$ and, by [Axiom 2\(i\)](#), satisfies $(\sigma, f_{\sigma^{-1}(s)} h) \sim (\sigma, c_s^\sigma(f)_{\sigma^{-1}(s)} h)$ for every $h \in F$;
- (ii) $\succsim_s^\sigma$ is a weak order represented by $c_s^\sigma(\cdot)$, and $f \sim_s^\sigma c_s^\sigma(f)$ for every $f \in F$.

Thus $c_s^\sigma(f)$ is well-defined and is the conditional certainty equivalent of the act f when the realization of the signal σ is s .

The interim preferences make available conditional analogs of the static notions introduced above; they are obtained by replacing the static relation with $\succsim_s^\sigma$ throughout. Say the event $E \cap \sigma^{-1}(s)$, for $E \in \mathcal{E}$, is *conditionally null at* (σ, s) if

$$x_{E \cap \sigma^{-1}(s)} y \sim_s^\sigma y \quad \text{for all } x, y \in X,$$

and say an arbitrary event A is conditionally null at (σ, s) if $A \cap \sigma^{-1}(s) \subseteq E \cap \sigma^{-1}(s)$ for some conditionally null $E \cap \sigma^{-1}(s)$. Let $\mathcal{N}_s^\sigma$ denote the collection of conditionally null events. For each event $A \subseteq \sigma^{-1}(s)$ with $A \notin \mathcal{N}_s^\sigma$ and each act $f \in F$, set, in exact parallel with the static bounds,

$$\begin{aligned} \underline{x}_s^\sigma(f_A) &:= \sup\{y \in X : \{\omega \in A : f(\omega) < y\} \in \mathcal{N}_s^\sigma\}, \\ \bar{x}_s^\sigma(f_A) &:= \inf\{z \in X : \{\omega \in A : f(\omega) > z\} \in \mathcal{N}_s^\sigma\}. \end{aligned}$$

I interpret $\underline{x}_s^\sigma(f_A)$ (respectively, $\bar{x}_s^\sigma(f_A)$) as the greatest lower (respectively, least upper) bound for the outcomes the decision-maker conceives, *interim*, as possibly arising on the restriction of f to A .

The next axiom requires that the same interval utility u that represents static EUU also govern the evaluation of uncertain outcomes at every history, so that information affects evaluation only through the belief component and not through the decision-maker's attitude to risk and ambiguity. In particular, an act whose conceivable outcomes, conditional on a realization s , cannot be further narrowed on any conditionally non-null ideal subevent of $E \cap \sigma^{-1}(s)$ is compared with sure outcomes in the same way as a diffuse act with the corresponding range of conceivable outcomes is compared with sure outcomes ex ante. Since this range is characterized by the interim lower and upper bounds defined above, the axiom compares these bounds with those of a diffuse benchmark. The comparisons are coordinatewise, corresponding to the monotonicity of the interval utility.

Axiom 3 (Invariant Uncertain Outcome Preference). Fix a signal $\sigma \in \Sigma$, a non-null realization $s \in \sigma(\Omega)^+$, and an ideal event $E \in \mathcal{E}$. Let $f, g \in F$ be acts, $h \in \mathcal{H}$ a diffuse act, and $z \in X$ an outcome. Write $A := E \cap \sigma^{-1}(s)$.

(i) Suppose $A \notin \mathcal{N}_s^\sigma$ and, for every ideal event $E' \in \mathcal{E}$ with $E' \cap A \notin \mathcal{N}_s^\sigma$,

$$\underline{x}_s^\sigma(f_{E' \cap A}) \leq \underline{x}(h) \quad \text{and} \quad \bar{x}_s^\sigma(f_{E' \cap A}) \leq \bar{x}(h).$$

Then $(\mathbf{0}, z) \succ (\mathbf{0}, h) \implies (\sigma, z_A g) \succ (\sigma, f_A g)$.

(ii) Suppose $A \notin \mathcal{N}_s^\sigma$ and, for every ideal event $E' \in \mathcal{E}$ with $E' \cap A \notin \mathcal{N}_s^\sigma$,

$$\underline{x}_s^\sigma(f_{E' \cap A}) \geq \underline{x}(h) \quad \text{and} \quad \bar{x}_s^\sigma(f_{E' \cap A}) \geq \bar{x}(h).$$

Then $(\mathbf{0}, h) \succ (\mathbf{0}, z) \implies (\sigma, f_A g) \succ (\sigma, z_A g)$.

(iii) If $A \in \mathcal{N}_s^\sigma$, then $(\sigma, f_A g) \sim (\sigma, z_A g)$.

When the hypotheses of (i) and (ii) hold simultaneously, the conditional lower and upper bounds of f on every conditionally non-null ideal subevent of A coincide with the corresponding bounds of h . The two parts of the axiom then imply that replacing f on A by the sure outcome $z^* \in X$ satisfying

$$v(z^*) = u(\underline{x}(h), \bar{x}(h))$$

is a matter of indifference, irrespective of the act specified outside A . This is exactly the rate at which the static preferences exchange a diffuse prospect with those conceivable lower and upper outcomes for a sure outcome.

Axiom 3 fixes only the utility side of interim evaluation. Information may change the agent's beliefs, but it may not change the interval utility u that represents his attitude toward risk and ambiguity. What remains open is which conditional belief over conditionally ideal events should govern interim rankings when the realized information cell $\sigma^{-1}(s)$ is not itself ideal. **Axiom 4** resolves that remaining indeterminacy by fixing the belief-updating component: interim rankings of conditionally ideal positions must agree with the ex ante rankings of the corresponding positions transplanted onto the uniformly assigned pieces generated by the uniform refinement construction.

To motivate this updating rule, fix a signal $\sigma \in \Sigma$ and consider its associated ideal split $\{E_Q^\sigma \in \mathcal{E} : Q \subseteq \sigma(\Omega), Q \neq \emptyset\}$. Recall that for any $Q \subseteq \sigma(\Omega)$ and any ideal event $E \in \mathcal{E}$ with $E \cap E_Q^\sigma \notin \mathcal{N}$, the family

$$\{E \cap E_Q^\sigma \cap \sigma^{-1}(s) : s \in Q\}$$

forms a $|Q|$ -element partition of the ideal event $E \cap E_Q^\sigma$. Moreover, because $\succeq$ evaluates any pair $(\mathbf{0}, f)$ through the $\mathcal{E}$ -measurable bounds of f , the REUU maximizer views the elements of this partition as ex ante exchangeable: for any $s', s'' \in Q$, any $x > y$, and any act f , he is indifferent between betting on (or against) $E \cap E_Q^\sigma \cap \sigma^{-1}(s')$ and betting on (or against) $E \cap E_Q^\sigma \cap \sigma^{-1}(s'')$. Intuitively, on the residual ideal cell E_Q^σ the signal cannot be further ideal-refined ex ante, so the singletons $s \in Q$ are treated symmetrically.

To respect this exchangeability in the construction of realization-contingent beliefs, the REUU maximizer allocates the prior weight $\mu(E \cap E_Q^\sigma)$ uniformly across the $|Q|$ pieces $\{E \cap E_Q^\sigma \cap \sigma^{-1}(s) : s \in Q\}$. The next definition introduces the corresponding uniform refinement partition, indexed by a finite ideal partition so that conditionally ideal positions, and not merely bets, can be transplanted.

Definition 7 (Uniform refinement ideal partition). Fix a signal $\sigma \in \Sigma$, with associated ideal split $\{E_Q^\sigma \in \mathcal{E} : Q \subseteq \sigma(\Omega), Q \neq \emptyset\}$, and a finite ideal partition $\mathbb{E} = \{E_1, \dots, E_k\}$ of Ω . For each non-empty $Q \subseteq \sigma(\Omega)$ choose an ideal partition $\{E_{Q,s}^{\sigma, \mathbb{E}} : s \in Q\}$ of E_Q^σ such that, for every $s \in Q$ and every $i = 1, \dots, k$,

$$\mu(E_{Q,s}^{\sigma, \mathbb{E}} \cap E_i) = \frac{\mu(E_Q^\sigma \cap E_i)}{|Q|}.$$

Define the induced uniform refinement partition $\{E_s : s \in \sigma(\Omega)\}$ by

$$E_s := \bigcup_{\substack{Q \subseteq \sigma(\Omega) \\ s \in Q}} E_{Q,s}^{\sigma, \mathbb{E}}, \quad s \in \sigma(\Omega).$$

Such choices exist: since μ is non-atomic and countably additive, the $\mathbb{R}^k$ -valued measure $A \mapsto (\mu(A \cap E_1), \dots, \mu(A \cap E_k))$ on the ideal subsets of E_Q^σ is non-atomic, so by **Lyapunov's (1940)** convexity theorem, iterating on successive remainders, each E_Q^σ can be split into $|Q|$ ideal events of equal vector mass. By construction,

$$\mu(E_i \cap E_s) = \sum_{\substack{Q \subseteq \sigma(\Omega) \\ s \in Q}} \frac{\mu(E_Q^\sigma \cap E_i)}{|Q|} \quad (i = 1, \dots, k)$$

$$\text{and } \mu(E_s) = \sum_{\substack{Q \subseteq \sigma(\Omega) \\ s \in Q}} \frac{\mu(E_Q^\sigma)}{|Q|} = p_s^\sigma,$$

the latter obtained by summing the former over i ; neither depends on the choices made. Taking $\mathbb{E} = \{E, \Omega \setminus E\}$ recovers the uniform refinement associated with a single ideal event E .

The last axiom formalizes this updating rule: a conditionally ideal position built from a finite ideal partition is ranked against a sure outcome, interim, exactly as its uniformly refined transplant is ranked against the corresponding transplanted sure outcome ex ante.

Axiom 4 (Invariant Conditional Betting Preferences). Fix a signal $\sigma \in \Sigma$ and a finite ideal partition $\mathbb{E} = \{E_1, \dots, E_k\}$ of Ω . Let $\{E_s : s \in \sigma(\Omega)\}$ be a uniform refinement ideal partition associated with $(\sigma, \mathbb{E})$. For each $s \in \sigma(\Omega)^+$ and all outcomes $x_1, \dots, x_k, x \in X$:

$$g \succeq_s^\sigma x \iff (\mathbf{0}, g_{E_s} \ell) \succeq (\mathbf{0}, x_{E_s} \ell),$$

where g denotes the (ideal) act taking the value x_i on E_i for each $i = 1, \dots, k$.

Restricting attention to two-element partitions $\{E, \Omega \setminus E\}$ with stakes m on E and ℓ on its complement, **Axiom 4** says interim rankings of bets on conditionally ideal events must respect the ex ante exchangeability encoded by the uniform refinement construction; the general form extends the same invariance from bets to conditionally ideal positions. Since the ex ante ranking on the right

side depends only on the measures $\mu(E_i \cap E_s)$, which [Definition 7](#) pins down, the content of the axiom does not depend on which admissible uniform refinement partition is chosen.

The representation theorem follows.

Theorem 1. Suppose [Assumption 1](#) holds and at least one diffuse event exists. Then the binary relation $\succsim$ is an REUU preference, represented as in [Definition 5](#) by the pair $\langle \mu, u \rangle$ of [Assumption 1](#), if and only if [Axioms 1 – 4](#) hold.

[Theorem 1](#) characterizes the ex ante relation, and the interim preferences come with it. The proof of sufficiency establishes that, under [Axioms 1 – 4](#), for every signal $\sigma \in \Sigma$ and every non-null realization $s \in \sigma(\Omega)^+$, the interim preference $\succsim_s^\sigma$ is represented by the conditional functional V_s^σ of [Definition 5](#). An REUU preference thus carries, at each non-null realization of each signal, an interim preference of EUU form with respect to the conditional belief μ_s^σ : the ex ante representation and the family of interim representations are two descriptions of the same preference.

Remark 4. Under the conditions of [Theorem 1](#), for every signal σ , every $s \in \sigma(\Omega)^+$ and every $E \in \mathcal{E}$, the event $E \cap \sigma^{-1}(s)$ is ideal with respect to the interim preference $\succsim_s^\sigma$ (in the sense of [Definition 6](#) applied to $\succsim_s^\sigma$): interim rankings of acts adapted to $\mathcal{E}_s^\sigma$ admit a conditional expected-utility evaluation, from which both **P2** implications follow by direct computation. By the same computation, for an ideal signal σ the relation $\succsim_s^\sigma$ coincides with the relation defined in [\(3\)](#): when $\sigma^{-1}(s)$ is ideal, the map $c \mapsto V(\mathbf{0}, c_{\sigma^{-1}(s)} k)$ is strictly increasing for every act k , since $\mu([\sigma^{-1}(s)]_*) = \mu(\sigma^{-1}(s)) > 0$.

4. Neo-additive signal-acts

This section identifies a tractable subclass of signal-acts, *neo-additive signal-acts*, for which both ex ante and interim evaluations admit a parsimonious “neo-EU” representation. The representation parallels the neoadditive capacity model of [Chateauneuf et al. \(2007\)](#), but in REUU the interval utility u is unrestricted. When u is nonseparable, the resulting ambiguity term can generate Machina-style reflection patterns (violations of tail separability) that are impossible under any CEU model, including neoadditive CEU.

4.1. Neo-additive acts and signal-acts

Definition 8 (Neo-additive acts). The act $g \in F$ is *neo-additive with respect to the ideal event* $G \in \mathcal{E}$ if for all $x \geq y$ such that $\mu_*(\{\omega \in \Omega : y \leq g(\omega) \leq x\}) \in (0, 1)$, where the equalities below are understood up to μ -null events,

1. $[\{\omega \in \Omega : y \leq g(\omega) \leq x\}]_* = \{\omega \in \Omega : y \leq g(\omega) \leq x\} \cap G$; and
2. $\Omega \setminus [\Omega \setminus \{\omega \in \Omega : y \leq g(\omega) \leq x\}]_* = \{\omega \in \Omega : y \leq g(\omega) \leq x\} \cup (\Omega \setminus G)$.

An act is *neo-additive* if it is neoadditive with respect to some $G \in \mathcal{E}$.

Neo-additivity with respect to G means: on the “ideal” region G the act behaves like an ideal act, while on the complement $\Omega \setminus G$ it is evaluated only through its worst and best conceivable prizes $\underline{x}(g)$ and $\bar{x}(g)$. In particular, the greatest lower and least upper *ideal* bounds of g are $g_G \underline{x}(g)$ and $g_G \bar{x}(g)$.

Definition 9 (Neo-additive signal-acts). A signal-act $(\tau, g) \in \Sigma \times F$ is *neo-additive* if there exists $G \in \mathcal{E}$ such that:

1. g is neoadditive with respect to G ; and
2. the ideal split of τ satisfies

$$G \setminus \left(\bigcup_{s \in \tau(\Omega)^+} E_{\{s\}}^\tau \right) \in \mathcal{N} \quad \text{and} \quad (\Omega \setminus G) \setminus E_{\tau(\Omega)^+}^\tau \in \mathcal{N}.$$

For the null signal $\mathbf{0}$, $\mathbf{0}(\Omega)^+ = \{o\}$ and $E_{\{o\}}^\mathbf{0} = E_{\mathbf{0}(\Omega)^+}^\mathbf{0} = \Omega$, so condition 2 is automatic; hence $(\mathbf{0}, g)$ is neoadditive whenever g is neoadditive. When $|\tau(\Omega)^+| > 1$, condition 2 says (up to null events) that the singleton elements of the ideal split account for G , while the residual element $E_{\tau(\Omega)^+}^\tau$ accounts for $\Omega \setminus G$; in particular, $E_Q^\tau \in \mathcal{N}$ for all $Q \subset \tau(\Omega)^+$ with $|Q| > 1$.

4.2. Evaluating neoadditive signal-acts

Fix g neoadditive with respect to $G \in \mathcal{E}$. Writing $v(x) := u(x, x)$ and setting $\lambda := 1 - \mu(G)$, the static EUU of $(\mathbf{0}, g)$ is

$$\begin{aligned} V(\mathbf{0}, g) &= \int u\left(\underline{g}^\mu, \bar{g}^\mu\right) d\mu \\ &= \mu(G) \int v(g) d\pi + (1 - \mu(G)) u(\underline{x}(g), \bar{x}(g)) \\ &= (1 - \lambda) \int_X v dF_g^\pi + \lambda u(\underline{x}(g), \bar{x}(g)), \end{aligned} \tag{8}$$

where (when $\mu(G) > 0$) π is the conditional probability on G ,

$$\pi(B) := \frac{\mu(B \cap G)}{\mu(G)} \quad \text{for } B \in \mathcal{E}^\pi := \{B : B \cap G \in \mathcal{E}\}, \quad (9)$$

and $F_g^\pi(x) := \pi(\{\omega : g(\omega) \leq x\})$ is the π -CDF of g . In the separable special case $u(x, y) = \alpha v(x) + (1 - \alpha)v(y)$, (8) reduces to the Chateaufeuf et al. neoadditive CEU formula; then the “tail increment” $u(a, m) - u(a, x) = (1 - \alpha)(v(m) - v(x))$ is independent of the lower bound a . With nonseparable u , this independence can fail.

Example 3 (Neo-additive tail nonseparability). Pick prizes $\ell < z < x < m$. Fix $G \in \mathcal{E}$ with $\mu(G) = 1 - \lambda \in (0, 1)$ and define π as in (9). Pick an ideal event $E \in \mathcal{E}$ with $E \subseteq G$ and $\pi(E) = \frac{1}{2}$. Fix a diffuse event $D \in \mathcal{D}$ such that both $(\Omega \setminus G) \cap D$ and $(\Omega \setminus G) \cap (\Omega \setminus D)$ are nonnull.

Define four neoadditive acts $f, g, \tilde{f}, \tilde{g}$ by their restrictions to G and $\Omega \setminus G$:

$$f|_G = \begin{cases} x, & \omega \in E, \\ z, & \omega \in G \setminus E, \end{cases} \quad g|_G \equiv z, \quad \tilde{f}|_G \equiv z, \quad \tilde{g}|_G = \begin{cases} x, & \omega \in E, \\ z, & \omega \in G \setminus E, \end{cases}$$

and on the diffuse part

$$f|_{\Omega \setminus G} = \begin{cases} \ell, & \omega \in (\Omega \setminus G) \cap D, \\ m, & \omega \in (\Omega \setminus G) \cap (\Omega \setminus D), \end{cases} \quad g|_{\Omega \setminus G} = \begin{cases} \ell, & \omega \in (\Omega \setminus G) \cap D, \\ x, & \omega \in (\Omega \setminus G) \cap (\Omega \setminus D), \end{cases}$$

$$\tilde{f}|_{\Omega \setminus G} = \begin{cases} z, & \omega \in (\Omega \setminus G) \cap D, \\ m, & \omega \in (\Omega \setminus G) \cap (\Omega \setminus D), \end{cases} \quad \tilde{g}|_{\Omega \setminus G} = \begin{cases} z, & \omega \in (\Omega \setminus G) \cap D, \\ x, & \omega \in (\Omega \setminus G) \cap (\Omega \setminus D). \end{cases}$$

Then

$$(\underline{x}(f), \bar{x}(f)) = (\ell, m), \quad (\underline{x}(g), \bar{x}(g)) = (\ell, x),$$

$$(\underline{x}(\tilde{f}), \bar{x}(\tilde{f})) = (z, m), \quad (\underline{x}(\tilde{g}), \bar{x}(\tilde{g})) = (z, x).$$

Let $\Delta := \frac{1-\lambda}{2}(v(x) - v(z)) > 0$. Using (8) and canceling common terms,

$$V(\mathbf{0}, f) - V(\mathbf{0}, g) = \Delta + \lambda(u(\ell, m) - u(\ell, x)),$$

$$V(\mathbf{0}, \tilde{f}) - V(\mathbf{0}, \tilde{g}) = -\Delta + \lambda(u(z, m) - u(z, x)).$$

Hence one obtains Machina's (2009) reflection pattern $(\mathbf{0}, f) > (\mathbf{0}, g)$ and $(\mathbf{0}, \tilde{g}) > (\mathbf{0}, \tilde{f})$ whenever

$$u(\ell, m) - u(\ell, x) > \Delta/\lambda > u(z, m) - u(z, x). \quad (10)$$

Condition (10) is incompatible with the separable neoadditive CEU benchmark, in which the increment $u(a, m) - u(a, x)$ is independent of a .

Now consider a neoadditive signal-act (τ, g) with $|\tau(\Omega)^+| > 1$. For each $s \in \tau(\Omega)^+$, the pseudo-prior (Definition 4) simplifies to

$$p_s^\tau = \mu(G \cap \tau^{-1}(s)) + \frac{\mu(\Omega \setminus G)}{|\tau(\Omega)^+|} = (1 - \lambda)\pi(\tau^{-1}(s)) + \lambda \frac{1}{|\tau(\Omega)^+|}.$$

Define

$$\pi_s^\tau(B) := \begin{cases} \frac{\pi(B \cap \tau^{-1}(s))}{\pi(\tau^{-1}(s))}, & \text{if } \pi(\tau^{-1}(s)) > 0, \\ \pi(B), & \text{if } \pi(\tau^{-1}(s)) = 0, \end{cases} \quad (B \in \mathcal{E}^\pi),$$

where the second line is an immaterial convention for a π -null information cell. Then

$$\mu_s^\tau(E \cap \tau^{-1}(s) \cap G) = \frac{\mu(E \cap \tau^{-1}(s) \cap G)}{p_s^\tau} = (1 - \lambda_s^\tau) \pi_s^\tau(E) \quad (E \in \mathcal{E}^\pi),$$

where the updated lack-of-confidence parameter is

$$\lambda_s^\tau := \frac{\lambda/|\tau(\Omega)^+|}{(1 - \lambda)\pi(\tau^{-1}(s)) + \lambda/|\tau(\Omega)^+|}. \quad (11)$$

If $\pi(\tau^{-1}(s)) = 0$, then $\lambda_s^\tau = 1$, so the expected-utility component receives zero weight and the convention for π_s^τ is irrelevant. Consequently, the interim preferences $\succsim_s^\tau$ admit a conditional neoEU representation:

$$V_s^\tau(g) = (1 - \lambda_s^\tau) \int_X v dF_g^{\pi_s^\tau} + \lambda_s^\tau u(\underline{x}(g_{\tau^{-1}(s)}), \bar{x}(g_{\tau^{-1}(s)})), \quad (12)$$

where $F_g^{\pi_s^\tau}(x) := \pi_s^\tau(\{\omega : g(\omega) \leq x\})$.

Remark 5. From (11): (i) if $|\tau(\Omega)^+| = 1$ (only one non-null realization), then $\lambda_s^\tau = \lambda$; (ii) if $0 < \lambda < 1$ and $|\tau(\Omega)^+| > 1$, then $\lambda_s^\tau < \lambda$ iff $\pi(\tau^{-1}(s)) > 1/|\tau(\Omega)^+|$ (and conversely); (iii) if $\pi(\tau^{-1}(s)) = 0$, then $\lambda_s^\tau = 1$ (all confidence in π is lost on an event π deems impossible).

4.3. Neo-additive sources

Definitions 8–9 characterize when a given act or signal-act is neo-additive. Since every ideal act is neoadditive (take $G = \Omega$), it is natural to ask whether there are priors π for which *all* π -measurable acts and $\mathcal{E}^\pi$ -adapted signal-acts inherit a neoEU representation.

Definition 10 (Neo-additive source). A prior $\pi \in \Pi$ (with domain $\mathcal{E}^\pi$) is a *neo-additive source* if there exists a non-null $G \in \mathcal{E} \setminus \mathcal{N}$ such that

$$\mu_*(A) = \begin{cases} \mu(G)\pi(A), & \text{if } \pi(A) < 1, \\ 1, & \text{if } \pi(A) = 1, \end{cases} \quad \text{for all } A \in \mathcal{E}^\pi.$$

The term “source” is motivated by the fact that for all $A, B \in \mathcal{E}^\pi$ with $\pi(A), \pi(B) \in (0, 1)$ and all $x > y$,

$$V(\mathbf{0}, x_A y) \geq V(\mathbf{0}, x_B y) \iff \pi(A) \geq \pi(B),$$

matching the defining feature of sources in [Gul and Pesendorfer \(2015\)](#).

Notice first that [Definition 10](#) implies that every π -null event is μ -null. Indeed, if $\pi(A) = 0$, then $\pi(A^c) = 1$ and hence $\mu_*(A^c) = 1$. Since the inner-measure supremum is attained and μ is complete, A is an ideal null event.

For any CDF F write

$$\underline{x}(F) := \inf\{y \in X : F(y) > 0\}, \quad \bar{x}(F) := \inf\{z \in X : F(z) = 1\}.$$

Suppose the prior π is a neoadditive source. Then for any $g \in F^\pi$,

$$V(\mathbf{0}, g) = (1 - \lambda) \int_X v dF_g^\pi + \lambda u(\underline{x}(F_g^\pi), \bar{x}(F_g^\pi)), \quad (13)$$

with $\lambda = 1 - \mu(G)$. Moreover, if (τ, g) is such that both τ and g are adapted to $\mathcal{E}^\pi$, then for each $s \in \tau(\Omega)^+$,

$$V_s^\tau(g) = (1 - \lambda_s^\tau) \int_X v dF_g^{\pi_s^\tau} + \lambda_s^\tau u(\underline{x}(F_g^{\pi_s^\tau}), \bar{x}(F_g^{\pi_s^\tau})), \quad (14)$$

where π_s^τ is the Bayesian update of π on $\tau^{-1}(s)$ and λ_s^τ is given by [\(11\)](#). The update is well-defined: if $\pi(\tau^{-1}(s)) = 0$, then the preceding observation would make $\tau^{-1}(s)$ μ -null, contrary to $s \in \tau(\Omega)^+$. Hence both the static *and* the interim preferences are probabilistically sophisticated in the sense of [Machina and Schmeidler \(1992\)](#).

For an ideal event $H \in \mathcal{E}$ with $\mu(H) > 0$, call an event $D \subseteq H$ *diffuse relative to H* if

$$D \cap E \neq \emptyset \neq (H \setminus D) \cap E$$

for every non-null ideal event $E \subseteq H$; equivalently, neither D nor $H \setminus D$ contains a non-null ideal subset of H .

The next example gives a concrete interpretation of [definition 10](#): G is the region on which probabilities are trusted (evaluation proceeds via the induced π -distribution), whereas $\Omega \setminus G$ contributes only through the extreme bounds term $u(\underline{x}, \bar{x})$.

Example 4 (A canonical neoadditive source). Take $\Omega = [0, 1]^2$ and let L denote Lebesgue measure on $[0, 1]$. Let

$$\mathcal{E}_0 := \{[0, 1] \times B : B \in \mathcal{B}([0, 1])\}, \quad \mu_0([0, 1] \times B) := L(B),$$

and let $(\mathcal{E}, \mu)$ be the completion of $(\mathcal{E}_0, \mu_0)$. Fix $\bar{y} \in (0, 1)$, set $G = [0, 1] \times [0, \bar{y}]$ and $H := \Omega \setminus G$, and let $\{C_n\}_{n \geq 1}$ be a countably infinite partition of $[0, 1]$ into nonempty sets (for example, $C_1 = \{0\} \cup (1/2, 1]$ and $C_n = (1/(n+1), 1/n]$ for $n \geq 2$). Then $D_n := C_n \times (\bar{y}, 1]$ is diffuse relative to H for every n .

Let $\{q_n\}_{n \geq 1}$ enumerate the rationals in $[0, 1)$ and define $Z : \Omega \rightarrow [0, 1]$ by

$$Z(x, y) := \begin{cases} y/\bar{y}, & y \leq \bar{y}, \\ ((y - \bar{y})/(1 - \bar{y}) + q_n) \bmod 1, & (x, y) \in D_n. \end{cases}$$

Set

$$\mathcal{A}_0 := \{Z^{-1}(B) : B \in \mathcal{B}([0, 1])\}, \quad \pi_0(Z^{-1}(B)) := L(B),$$

and let $(\mathcal{E}^\pi, \pi)$ be the completion of $(\mathcal{A}_0, \pi_0)$. For $A = Z^{-1}(B)$,

$$A \cap G = [0, 1] \times \{\bar{y}t : t \in B\}, \quad \frac{\mu(A \cap G)}{\mu(G)} = L(B) = \pi_0(A).$$

If $0 < L(B) < 1$, the rational-shift argument used in the proof of [Proposition 1](#) shows that $A \cap H$ is diffuse relative to H ; if $L(B) = 0$, then A is contained in an ideal μ -null event, and if $L(B) = 1$ the same is true of A^c . It follows that

$$\mu_*(A) = \begin{cases} \mu(G)\pi_0(A), & \pi_0(A) < 1, \\ 1, & \pi_0(A) = 1. \end{cases}$$

The probability π_0 is non-atomic, and completion preserves non-atomicity. Every π_0 -null event is also μ -null, so both displayed identities are preserved by completion. Thus $\pi \in \Pi$ is a neoadditive source associated with G .

Example 4 gives an explicit construction without any set-theoretic assumption. The following proposition identifies the exact general condition under which such a source exists.

Proposition 1. Fix a non-null ideal event $G \in \mathcal{E} \setminus \mathcal{N}$. If $\mu(G) = 1$, then μ is a neoadditive source associated with G and satisfies

$$\mu(A) = \frac{\mu(A \cap G)}{\mu(G)} \quad \text{for every } A \in \mathcal{E}.$$

If $0 < \mu(G) < 1$, then there exists a neoadditive source $\pi \in \Pi$, with domain $\mathcal{E}^\pi$, satisfying

$$\pi(A) = \frac{\mu(A \cap G)}{\mu(G)} \quad \text{for every } A \in \mathcal{E}^\pi$$

if and only if $\Omega \setminus G$ admits a countably infinite partition $\{D_n\}_{n \geq 1}$ into events diffuse relative to $\Omega \setminus G$.

Remark 6. Under the continuum hypothesis, the condition in Proposition 1 is automatic: the diffuse-set construction of Gul and Pesendorfer (2014), as used in Appendix A.1 of Gul and Pesendorfer (2015), yields a countably infinite partition into diffuse events when applied to the normalized restriction of μ to $\Omega \setminus G$. Hence the existence conclusion holds for every non-null ideal G .

5. Related literature

For dynamic decision problems with non-additive beliefs and non-neutral attitudes toward ambiguity, two broad strands of work have developed.

A first, largely *statistical* strand studies updating rules for non-additive objects (capacities, sets of priors, belief functions) and the properties of the induced conditional expectations. Representative contributions include (Denneberg, 1994,?, 2002; Jaffray, 1992; Lehrer, 2005; Walley, 1991; Shafer, 1976; Kast et al., 2012), and more recently (Gul and Pesendorfer, 2021; Cerreia-Vioglio et al., 2024).

A second, *decision-theoretic* strand derives dynamic representations and updating rules from axioms imposed on ex ante and interim preferences; non-exhaustive examples include (Gilboa and Schmeidler, 1993; Sarin and Wakker, 1998; Ghirardato, 2002; Pires, 2002; Epstein and Schneider, 2003; Wang, 2003; Hanany and Klbanoff, 2007; Eichberger et al., 2007; Siniscalchi, 2011).

The present paper follows the decision-theoretic approach. It extends Gul and Pesendorfer's (2014) static expected uncertain utility (EUU) to *information decision problems* by introducing a conditional belief system (Definition 4) that specifies pseudo-priors and posteriors for *arbitrary* signals. The central feature is that, after any realized signal, the induced interim preferences retain an EUU form with the *same* interval utility u (model consistency), are consequentialist in the within-event sense used here, and are dynamically consistent in the recursive sense: optimizing after each realized signal yields an ex ante optimal plan. This contrasts with approaches that achieve dynamic consistency by relaxing consequentialism (e.g. Sarin and Wakker 1998,?, Siniscalchi (2011)) or with multiple-prior models that retain both consequentialism and dynamic consistency only under structural restrictions such as rectangularity (e.g. Epstein and Schneider 2003,?). The stronger property those models secure, that every ex ante optimal plan is interim optimal, holds here for ideal signals.

The *interpretation* of Definition 4 is related in spirit to “extended Bayesianism” (Piermont, 2021): when a signal is not adapted to ideal events, learning its realization changes the agent's ability to refine uncertainty *within* the realized information cell. Although the agent's awareness does not expand, the distinction between ideal (quantifiable) and diffuse (non-quantifiable) components is effectively sharpened upon learning. The conditional belief system then specifies how the prior ideal content is mapped into interim beliefs over the relative σ -algebra associated with the realized cell.

A natural benchmark for comparison is Ghirardato (2002), who shows that combining certain notions of consequentialism and dynamic consistency can force ex ante preferences back to subjective expected utility. The reconciliation is that the interim preference family induced here is *signal-dependent*: even for a fixed realization s , the interim functional $V_s^\sigma(\cdot)$ may depend on the entire signal σ and not merely on the conditioning event $\sigma^{-1}(s)$. In this sense the present framework retains within-event consequentialism while eschewing the stronger signal-independence implicit in some tree-based formulations.

Finally, Gul and Pesendorfer (2021) develop a recursive and consequentialist theory of belief revision for two-stage uncertainty under non-additive beliefs. Their approach constructs, for each signal, a “proxy” object that renders the signal's cells unambiguous before applying Bayes-type conditioning within each cell. The present paper is closest in spirit to this proxy-then-condition logic, but differs in primitives and scope: the starting point here is EUU preferences represented by a convex-ranged prior μ and interval utility u , and the conditional belief system is derived so as to preserve EUU model consistency and the consequentialist/dynamic-consistency package on unrestricted signal/act domains. In applications, this delivers a tractable dynamic EUU extension without imposing a priori domain restrictions or rectangularity-type constraints.

Declaration of competing interest

I, Simon Grant, declare that I have no relevant or material financial interests that relate to the research described in this paper.

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Data availability

No data was used for the research described in the article.

Appendix A. Proofs

A.1. Proof of Theorem 1

Throughout this proof, fix the prior $\mu \in \Pi$ (with domain $\mathcal{E}$) and interval utility u given by [Assumption 1](#) and, without loss of generality, normalize $u(\ell, \ell) := 0$ and $u(m, m) := 1$. For a signal σ and a realization $s \in \sigma(\Omega)$, write $B_s := \sigma^{-1}(s)$. For any event $B \subseteq \Omega$, write $\mu^*(B) := 1 - \mu_*(\Omega \setminus B)$ for its outer measure and

$$R_B := \Omega \setminus ([B]_* \cup [\Omega \setminus B]_*)$$

for the residual ideal event of the pair $\{B, \Omega \setminus B\}$, so that $\mu(R_B) = \mu^*(B) - \mu_*(B)$.

Two standard facts about envelopes are used repeatedly without further comment. First, envelopes exist and are essentially unique with respect to any countably additive measure on any sub-sigma-algebra: the pointwise supremum of countably many measurable minorants of an act is again a measurable minorant, so an essential-supremum construction delivers a measurable minorant, maximal up to null modification, that lies below the act pointwise; dually for majorants. Second, envelopes localize: if $\{G_1, \dots, G_n\}$ is a measurable partition of the relevant domain, the envelope of an act agrees almost everywhere on each G_i with the envelope of its restriction to G_i computed in the trace sigma-algebra, since measurable minorants (majorants) can be glued along the partition.

Lemma 2. Suppose [Assumption 1](#) holds. Then:

(a) For every event $B \subset \Omega$ and all outcomes $\ell \leq c' < y_0 < c \leq m$:

$$\begin{aligned} V(\mathbf{0}, c_B y_0) - V(\mathbf{0}, c'_B y_0) &= \mu([B]_*) (v(c) - v(c')) \\ &\quad + \mu(R_B) (u(y_0, c) - u(c', y_0)). \end{aligned}$$

(b) An event B is null if and only if $\mu^*(B) = 0$; equivalently, if and only if $B \in \mathcal{E}$ and $\mu(B) = 0$. Moreover, if $B \notin \mathcal{N}$ the display in (a) is strictly positive.

(c) For every signal σ and every $s \in \sigma(\Omega)$: $p_s^\sigma > 0$ if and only if $B_s \notin \mathcal{N}$; that is, $\sigma(\Omega)^+ = \{s \in \sigma(\Omega) : p_s^\sigma > 0\}$.

(d) For every $Q \subseteq \sigma(\Omega)$ with $s \in Q$ and every non-null ideal event $F \subseteq E_Q^\sigma$: $F \cap B_s \notin \mathcal{N}$. Consequently, for $s \in \sigma(\Omega)^+$: if $A \subseteq B_s$ satisfies $\mu^*(A) = 0$ then A has μ_s^σ -outer measure zero; and if $\mu_s^\sigma(E \cap B_s) > 0$ then $E \cap B_s \notin \mathcal{N}$.

Proof. (a) Consider the three-element ideal partition $\{[B]_*, R_B, [\Omega \setminus B]_*\}$ and the act $c_B y_0$ (recall $c > y_0$). On $[B]_*$ the act is constant at c , so its envelope pair is (c, c) almost everywhere there; on $[\Omega \setminus B]_*$ it is constant at y_0 . On R_B , no non-null ideal $F \subseteq R_B$ satisfies $F \subseteq B$ or $F \subseteq \Omega \setminus B$ (otherwise the corresponding inner-sleeve would not be maximal); hence every measurable minorant of the act is at most y_0 , and every measurable majorant at least c , almost everywhere on R_B , and both bounds are attained by constants. So the envelope pair on R_B is (y_0, c) almost everywhere. The analogous computation for $c'_B y_0$ (recall $c' < y_0$) yields the pairs (c', c') on $[B]_*$, (c', y_0) on R_B , and (y_0, y_0) on $[\Omega \setminus B]_*$. Integrating u over the three cells and differencing gives the display.

(b) If $\mu^*(B) = 0$ then B is contained in a null ideal event, so by the completeness of $\mathcal{E}$ we have $B \in \mathcal{E}$ with $\mu(B) = 0$; modifying an act on B then leaves its envelopes unchanged almost everywhere, so $B \in \mathcal{N}$. Conversely suppose $\mu^*(B) > 0$, so $\mu([B]_*) + \mu(R_B) > 0$. In the display of (a), the first term is nonnegative, and strictly positive when $\mu([B]_*) > 0$ since v is strictly increasing; the second term is strictly positive when $\mu(R_B) > 0$, because $y_0 > c'$ and $c > y_0$ imply $u(y_0, c) > u(c', y_0)$ by the strict monotonicity of the interval utility. Hence $(\mathbf{0}, c_B y_0) \sim (\mathbf{0}, c'_B y_0)$, whereas were B null both signal-acts would be indifferent to $(\mathbf{0}, y_0)$. So $B \notin \mathcal{N}$ and the display is strictly positive.

(c) First observe, by induction on $|P|$, that for every non-empty $P \subseteq \sigma(\Omega)$ the ideal events $\{E_Q^\sigma : \emptyset \neq \hat{Q} \subseteq P\}$ partition $[\sigma^{-1}(P)]_*$ up to null events: E_P^σ is by construction $[\sigma^{-1}(P)]_*$ less the union of the E_Q^σ for $\hat{Q} \subsetneq P$, and for each $\hat{Q} \subsetneq P$ the ideal event $[\sigma^{-1}(\hat{Q})]_* \setminus [\sigma^{-1}(P)]_*$ is null by the maximality of inner-sleeves. Now suppose $p_s^\sigma = 0$, so $\mu(E_Q^\sigma) = 0$ for every $Q \ni s$. Since the ideal split partitions Ω , the union of the E_Q^σ with $Q \not\ni s$ has measure one, and each such $E_Q^\sigma \subseteq [\sigma^{-1}(Q)]_* \subseteq \Omega \setminus B_s$; hence $\mu_*(\Omega \setminus B_s) = 1$, that is, $\mu^*(B_s) = 0$, and $B_s \in \mathcal{N}$ by (b). Conversely suppose $B_s \in \mathcal{N}$ and fix $Q \ni s$, writing $Q' := Q \setminus \{s\}$. Adjoining a null event changes no inner measure, so $\mu_*(\sigma^{-1}(Q)) = \mu_*(\sigma^{-1}(Q') \cup B_s) = \mu_*(\sigma^{-1}(Q'))$. By the exhaustion observation, $[\sigma^{-1}(Q')]_*$ coincides up to null events with $\bigcup_{\emptyset \neq \hat{Q} \subseteq Q'} E_{\hat{Q}}^\sigma$, all of which are excised in the construction of E_Q^σ ; hence $\mu(E_Q^\sigma) = 0$. As $Q \ni s$ was arbitrary, $p_s^\sigma = 0$.

(d) Suppose $F \subseteq E_Q^\sigma$ is ideal and non-null yet $F \cap B_s \in \mathcal{N}$, so $F \cap B_s \subseteq N_0$ for some null ideal N_0 . Then $F \setminus N_0$ is a non-null ideal subset of $\sigma^{-1}(Q \setminus \{s\})$ that, up to null events, is disjoint from $[\sigma^{-1}(Q \setminus \{s\})]_*$ (which, by the exhaustion observation in (c), is

the union of the E_Q^σ for $\emptyset \neq \hat{Q} \subseteq Q \setminus \{s\}$, all disjoint from E_Q^σ). This contradicts the maximality of the inner-sleeve of $\sigma^{-1}(Q \setminus \{s\})$. For the consequences: if $\mu^*(A) = 0$ then $A \subseteq N_0$ for some null ideal N_0 , and $\mu_s^\sigma(N_0 \cap B_s) = \sum_{Q \ni s} \mu(N_0 \cap E_Q^\sigma) / (|Q| p_s^\sigma) = 0$; and if $\mu_s^\sigma(E \cap B_s) > 0$ then $\mu(E \cap E_Q^\sigma) > 0$ for some $Q \ni s$, so applying the first assertion to $F := E \cap E_Q^\sigma$ yields $E \cap E_Q^\sigma \cap B_s \notin \mathcal{N}$, and a fortiori $E \cap B_s \notin \mathcal{N}$. $\square$

A.1.1. Proof of Lemma 1

Fix σ and $s \in \sigma(\Omega)^+$, and suppose [Assumption 1](#) and [Axioms 1](#) and [2](#) hold.

Identification. Existence of an outcome c with $(\sigma, f_{B_s} y) \sim (\sigma, c_{B_s} y)$ for every $y \in X$ is the first clause of [Axiom 2](#)(i). Suppose $c > c'$ both have this property for the act f . By transitivity, $(\sigma, c_{B_s} y) \sim (\sigma, c'_{B_s} y)$ for every $y \in X$; take $y := y_0$ for some $y_0 \in (c', c)$. The acts $c_{B_s} y_0$ and $c'_{B_s} y_0$ are adapted to σ , so [Axiom 1](#) and the static representation give $V(\mathbf{0}, c_{B_s} y_0) = V(\mathbf{0}, c'_{B_s} y_0)$, contradicting [Lemma 2](#)(a)–(b) since $B_s \notin \mathcal{N}$. Hence the outcome is unique; call it $c_s^\sigma(f)$. The identifying property involves f only through the acts $f_{B_s} y$, so $c_s^\sigma(f)$ depends only on the restriction of f to B_s . Finally, the second clause of [Axiom 2](#)(i) extends the indifference to every continuation: $(\sigma, f_{B_s} h) \sim (\sigma, c_s^\sigma(f)_{B_s} h)$ for every $h \in F$.

Representation. If $c_s^\sigma(f) \geq c_s^\sigma(g)$ then for every h , monotonicity ([Assumption 1](#)) yields

$$(\sigma, f_{B_s} h) \sim (\sigma, c_s^\sigma(f)_{B_s} h) \succeq (\sigma, c_s^\sigma(g)_{B_s} h) \sim (\sigma, g_{B_s} h),$$

so $f \succeq_s^\sigma g$. If instead $c_s^\sigma(f) < c_s^\sigma(g)$, take $y_0 \in (c_s^\sigma(f), c_s^\sigma(g))$ and $h := y_0$; by [Axiom 1](#) and [Lemma 2](#)(a)–(b), $(\sigma, c_s^\sigma(g)_{B_s} y_0) > (\sigma, c_s^\sigma(f)_{B_s} y_0)$, whence $(\sigma, g_{B_s} y_0) > (\sigma, f_{B_s} y_0)$ and $f \not\succeq_s^\sigma g$. Thus $\succeq_s^\sigma$ is complete, hence a weak order, and is represented by $c_s^\sigma(\cdot)$; and $f \sim_s^\sigma c_s^\sigma(f)$ because the two acts share the same conditional certainty equivalent. $\square$

The representation argument also delivers a comparison principle invoked repeatedly below: for any two acts f, g and any $s \in \sigma(\Omega)^+$, if $(\sigma, f_{B_s} h) \succeq (\sigma, g_{B_s} h)$ for every $h \in F$, then $c_s^\sigma(f) \geq c_s^\sigma(g)$, for otherwise the choice $h := y_0 \in (c_s^\sigma(f), c_s^\sigma(g))$ produces a strict reversal.

A.1.2. Sufficiency of the axioms

Suppose [Assumption 1](#), the existence of a diffuse event $D \in \mathcal{D}$, and [Axioms 1 – 4](#) hold. Fix a signal $\sigma \in \Sigma$; let $\{E_s^\sigma\}$ denote its ideal split and $\langle (p_s^\sigma)_s, (\mu_s^\sigma)_s : p_s^\sigma > 0 \rangle$ its uniform conditional belief system. By [Lemma 2](#)(c), $\sigma(\Omega)^+ = \{s : p_s^\sigma > 0\}$. Each μ_s^σ is a countably additive probability measure on $\mathcal{E}_s^\sigma$ (countable additivity is inherited term by term from μ ; well-definedness is [Remark 1](#)). It is complete as well: if $A = E \cap B_s \in \mathcal{E}_s^\sigma$ has $\mu_s^\sigma(A) = 0$, then $\mu(E \cap E_Q^\sigma) = 0$ for every $Q \ni s$, so A is contained in the ideal μ -null event $\bigcup_{Q \ni s} (E \cap E_Q^\sigma)$; every subset $C \subseteq A$ therefore belongs to $\mathcal{E}$ by completeness of μ , and $C = C \cap B_s \in \mathcal{E}_s^\sigma$. It is also non-atomic: if $\mu_s^\sigma(E \cap B_s) > 0$ then $\mu(E \cap E_Q^\sigma) > 0$ for some $Q \ni s$, and splitting the ideal event $E \cap E_Q^\sigma$ into two ideal events of equal measure (by the convex-rangedness of μ) splits the trace $E \cap B_s$ into two traces of intermediate conditional measure. Hence the conditional envelopes of [Definition 2](#) exist and are essentially unique, and V_s^σ is well-defined.

Step 1: Conditional certainty equivalents and pasting. By [Axiom 2](#)(i) and [Lemma 1](#), for each $s \in \sigma(\Omega)^+$ every act f has a unique conditional certainty equivalent $c_s^\sigma(f)$, and $\succeq_s^\sigma$ is a weak order represented by $c_s^\sigma(\cdot)$. Enumerate $\sigma(\Omega) = \{s_1, \dots, s_K\}$ and define $f^0 := f$ and, recursively, $f^k := c_{s_k}^\sigma(f)_{B_{s_k}} f^{k-1}$ if $s_k \in \sigma(\Omega)^+$, and $f^k := \ell_{B_{s_k}} f^{k-1}$ otherwise. If $s_k \in \sigma(\Omega)^+$ then, since f^{k-1} agrees with f on B_{s_k} (the earlier steps modified other cells only), [Axiom 2](#)(i) yields $(\sigma, f^{k-1}) \sim (\sigma, f^k)$; if $s_k \notin \sigma(\Omega)^+$ then $B_{s_k} \in \mathcal{N}$ and [Axiom 2](#)(ii) yields the same indifference. By transitivity $(\sigma, f) \sim (\sigma, f^K)$, and f^K is precisely the σ -adapted act, call it f_σ^c , that takes the value $c_s^\sigma(f)$ on B_s for each $s \in \sigma(\Omega)^+$ and the value ℓ elsewhere. So [Axiom 1](#) gives

$$(\sigma, f) \sim (\mathbf{0}, f_\sigma^c). \quad (\text{A.1})$$

In particular every signal-act possesses a certainty equivalent: $V(\mathbf{0}, x) = v(x)$ is continuous with range $[0, 1] \ni V(\mathbf{0}, f_\sigma^c)$, so some outcome x satisfies $(\mathbf{0}, x) \sim (\mathbf{0}, f_\sigma^c) \sim (\sigma, f)$. In view of [\(A.1\)](#) and [Definition 5](#), it remains to show that for each $s \in \sigma(\Omega)^+$ and every $f \in F$,

$$v(c_s^\sigma(f)) = V_s^\sigma(f) = \int_{B_s} u(\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma}) d\mu_s^\sigma. \quad (\text{A.2})$$

Step 2: Conditional expected utility on conditionally ideal simple acts. Fix $s \in \sigma(\Omega)^+$, a finite ideal partition $\mathbb{E} = \{E_1, \dots, E_k\}$ of Ω , outcomes $x_1, \dots, x_k$, and let g denote the ideal act taking value x_i on E_i . Choose a uniform refinement ideal partition $\{E_t : t \in \sigma(\Omega)\}$ associated with $(\sigma, \mathbb{E})$ ([Definition 7](#); existence was established in the main text). The two acts on the right side of [Axiom 4](#) are ideal, so the static representation evaluates them by expected utility:

$$V(\mathbf{0}, g_{E_s} \ell) = \sum_{i=1}^k v(x_i) \mu(E_i \cap E_s) = p_s^\sigma \sum_{i=1}^k v(x_i) \mu_s^\sigma(E_i \cap B_s)$$

$$\text{and } V(\mathbf{0}, x_{E_s} \ell) = p_s^\sigma v(x),$$

using $\mu(E_i \cap E_s) = \sum_{Q \ni s} \mu(E_Q^\sigma \cap E_i) / |Q| = p_s^\sigma \mu_s^\sigma(E_i \cap B_s)$, which follows from [Definitions 4](#) and [7](#).

Since $c_s^\sigma(x) = x$ for every constant, [Axiom 4](#) and [Lemma 1](#) yield, for every $x \in X$,

$$c_s^\sigma(g) \geq x \iff \sum_{i=1}^k \mu_s^\sigma(E_i \cap B_s) v(x_i) \geq v(x).$$

Taking the supremum over x on both sides and using the continuity and strict monotonicity of v (the sum lies in $[v(\ell), v(m)]$, being a convex combination) gives

$$v(c_s^\sigma(g)) = \sum_{i=1}^k \mu_s^\sigma(E_i \cap B_s) v(x_i) = \int_{B_s} v \circ g \, d\mu_s^\sigma. \quad (\text{A.3})$$

Since c_s^σ depends only on restrictions to B_s , [\(A.3\)](#) extends to every act agreeing on B_s with an ideal simple act.

Step 3: Identification of conditionally null events and interim bounds. Let $E \in \mathcal{E}$. Applying [\(A.3\)](#) to the acts $x_{E \cap B_s} y$ (via the partition $\{E, \Omega \setminus E\}$) gives $v(c_s^\sigma(x_{E \cap B_s} y)) = \mu_s^\sigma(E \cap B_s) v(x) + (1 - \mu_s^\sigma(E \cap B_s)) v(y)$, and since v is strictly increasing,

$$E \cap B_s \text{ is conditionally null at } (\sigma, s) \iff \mu_s^\sigma(E \cap B_s) = 0. \quad (\text{A.4})$$

Hence $A \in \mathcal{N}_s^\sigma$ if and only if $A \cap B_s$ has μ_s^σ -outer measure zero, and for any act f and any $A \subseteq B_s$ with $A \notin \mathcal{N}_s^\sigma$, the interim bounds $\underline{x}_s^\sigma(f_A)$ and $\bar{x}_s^\sigma(f_A)$ coincide with the μ_s^σ -essential bounds of f on A : the greatest y such that $\{\omega \in A : f(\omega) < y\}$ has μ_s^σ -outer measure zero, and dually.

Step 4: Approximation by bounding simple envelopes. Fix $s \in \sigma(\Omega)^+$, an act f , and $\varepsilon > 0$. By the uniform continuity of u on its compact domain, choose $\delta > 0$ such that

$$|u(a, b) - u(a', b')| \leq \varepsilon$$

whenever $|a - a'| \leq 2\delta$ and $|b - b'| \leq 2\delta$. Choose a grid

$$\ell = t_0 < t_1 < \dots < t_M = m$$

with mesh less than δ .

Fix versions

$$\underline{f} := \underline{f}^{\mu_s^\sigma} \quad \text{and} \quad \bar{f} := \bar{f}^{\mu_s^\sigma}$$

of the conditional lower and upper envelopes of f , satisfying

$$\underline{f} \leq f \leq \bar{f}$$

pointwise on B_s . For $0 \leq i \leq j \leq M - 1$, define the conditionally ideal subevents

$$A_{ij} := \left\{ \omega \in B_s : \underline{f}(\omega) \in [t_i, t_{i+1}), \bar{f}(\omega) \in [t_j, t_{j+1}] \right\} \in \mathcal{E}_s^\sigma,$$

with the intervals containing m closed at their upper endpoint. The nonempty events A_{ij} form a finite partition of B_s .

Fix $A_{ij} \notin \mathcal{N}_s^\sigma$. I first show that, for every ideal event $E' \in \mathcal{E}$ satisfying

$$E' \cap A_{ij} \notin \mathcal{N}_s^\sigma,$$

or equivalently by Step 3,

$$\mu_s^\sigma(E' \cap A_{ij}) > 0,$$

we have

$$\underline{x}_s^\sigma(f_{E' \cap A_{ij}}) \in [t_i, t_{i+1}] \quad \text{and} \quad \bar{x}_s^\sigma(f_{E' \cap A_{ij}}) \in [t_j, t_{j+1}]. \quad (\text{A.5})$$

The inequalities

$$\underline{x}_s^\sigma(f_{E' \cap A_{ij}}) \geq t_i \quad \text{and} \quad \bar{x}_s^\sigma(f_{E' \cap A_{ij}}) \leq t_{j+1}$$

follow directly from

$$t_i \leq \underline{f} \leq f \leq \bar{f} \leq t_{j+1}$$

on A_{ij} .

To establish the remaining lower-bound inequality, suppose instead that

$$\{f < t_{i+1}\} \cap E' \cap A_{ij}$$

were conditionally null. It would then be contained in some $\tilde{N} \in \mathcal{E}_s^\sigma$ with $\mu_s^\sigma(\tilde{N}) = 0$. The pointwise maximum of $\underline{f}$ and the function that equals t_{i+1} on

$$(E' \cap A_{ij}) \setminus \tilde{N}$$

and ℓ elsewhere would then be an $\mathcal{E}_s^\sigma$ -measurable minorant of f that exceeds $\underline{f}$ on a set of positive μ_s^σ measure. This contradicts the maximality of $\underline{f}$. Hence

$$\underline{x}_s^\sigma(f_{E' \cap A_{ij}}) \leq t_{i+1}.$$

The inequality

$$\bar{x}_s^\sigma(f_{E' \cap A_{ij}}) \geq t_j$$

follows by the analogous argument using the minimality of $\bar{f}$.

Now construct two conditionally ideal simple acts. For each A_{ij} , set

$$z_{ij}^- := v^{-1}(u(t_i, t_j)) \quad \text{and} \quad z_{ij}^+ := v^{-1}(u(t_{i+1}, t_{j+1})).$$

The inverses exist since $u(x, y) \in [v(x), v(y)]$ for $x \leq y$, the interval utility being weakly increasing in each coordinate by its joint strict monotonicity and continuity.

Let ψ^- (respectively, ψ^+) denote the act taking the value z_{ij}^- (respectively, z_{ij}^+) on A_{ij} and agreeing with f off B_s ; if $A_{ij} \in \mathcal{N}_s^\sigma$, set both values equal to t_0 . Since the partition $\{A_{ij}\}$ admits ideal representatives whose traces on B_s are pairwise disjoint and exhaust B_s , the restrictions of $\psi^\pm$ to B_s agree with those of ideal simple acts.

Fix any $h \in \mathcal{F}$. I claim that

$$(\sigma, \psi_{B_s}^+ h) \succeq (\sigma, f_{B_s} h) \succeq (\sigma, \psi_{B_s}^- h). \quad (\text{A.6})$$

For the second inequality, replace f by ψ^- on the subevents A_{ij} one at a time and chain the resulting comparisons by transitivity. At each stage the current act agrees with f on every A_{ij} not yet replaced. It therefore suffices to establish, for each A_{ij} and every act g specified outside A_{ij} , that

$$(\sigma, f_{A_{ij}} g) \succeq (\sigma, (z_{ij}^-)_{A_{ij}} g).$$

If $A_{ij} \in \mathcal{N}_s^\sigma$, this follows directly from [Axiom 3\(iii\)](#).

Suppose instead that $A_{ij} \notin \mathcal{N}_s^\sigma$. Let

$$h^- := (t_j)_D t_i \in \mathcal{H},$$

a diffuse act with

$$\underline{x}(h^-) = t_i \quad \text{and} \quad \bar{x}(h^-) = t_j.$$

When $j = i$, take h^- to be the constant act t_i , which belongs trivially to $\mathcal{H}$. Then

$$V(\mathbf{0}, h^-) = u(t_i, t_j) = v(z_{ij}^-),$$

and hence

$$(\mathbf{0}, h^-) \sim (\mathbf{0}, z_{ij}^-).$$

By [\(A.5\)](#), for every ideal event E' satisfying $E' \cap A_{ij} \notin \mathcal{N}_s^\sigma$,

$$\underline{x}_s^\sigma(f_{E' \cap A_{ij}}) \geq t_i = \underline{x}(h^-)$$

and

$$\bar{x}_s^\sigma(f_{E' \cap A_{ij}}) \geq t_j = \bar{x}(h^-).$$

[Axiom 3\(ii\)](#) therefore gives

$$(\sigma, f_{A_{ij}} g) \succeq (\sigma, (z_{ij}^-)_{A_{ij}} g).$$

Chaining these comparisons over the finitely many elements of the partition establishes the second inequality in [\(A.6\)](#).

The first inequality is obtained symmetrically. For each non-null A_{ij} , let

$$h^+ := (t_{j+1})_D t_{i+1} \in \mathcal{H},$$

taking the constant act t_{i+1} when $j = i$. By [\(A.5\)](#),

$$\underline{x}_s^\sigma(f_{E' \cap A_{ij}}) \leq t_{i+1} = \underline{x}(h^+)$$

and

$$\bar{x}_s^\sigma(f_{E' \cap A_{ij}}) \leq t_{j+1} = \bar{x}(h^+)$$

for every relevant E' , while

$$(\mathbf{0}, z_{ij}^+) \sim (\mathbf{0}, h^+).$$

Axiom 3(i), together with **Axiom 3(iii)** on the conditionally null elements of the partition, therefore yields the first inequality in (A.6).

Since (A.6) holds for every h , the comparison principle recorded after the proof of **Lemma 1** gives

$$c_s^\sigma(\psi^-) \leq c_s^\sigma(f) \leq c_s^\sigma(\psi^+).$$

By (A.3),

$$v(c_s^\sigma(\psi^\pm)) = \sum_{ij} \mu_s^\sigma(A_{ij}) v(z_{ij}^\pm),$$

where the conditionally null A_{ij} may be omitted from the sum.

For every $\omega \in A_{ij}$, the pair

$$(\underline{f}(\omega), \bar{f}(\omega))$$

lies within the grid mesh, coordinate by coordinate, of both

$$(t_i, t_j) \quad \text{and} \quad (t_{i+1}, t_{j+1}).$$

Hence, by the choice of δ ,

$$|v(z_{ij}^\pm) - u(\underline{f}(\omega), \bar{f}(\omega))| \leq \varepsilon \quad \text{for every } \omega \in A_{ij}.$$

Integrating gives

$$|v(c_s^\sigma(\psi^\pm)) - V_s^\sigma(f)| \leq \varepsilon.$$

Combining these inequalities and letting $\varepsilon \downarrow 0$ establishes (A.2).

Step 5: The representation. By Steps 1 and 4, for every signal σ and act f , the act f_σ^c coincides with the act f_σ of **Definition 5**, and $(\sigma, f) \sim (\mathbf{0}, f_\sigma)$. Hence for any two signal-acts,

$$(\sigma, f) \succeq (\tau, g) \iff V(\mathbf{0}, f_\sigma) \geq V(\mathbf{0}, g_\tau) \iff V(\sigma, f) \geq V(\tau, g)$$

with V as in **Definition 5**; that is, $\succeq$ is an REUU preference represented by $\langle \mu, u \rangle$. (For $\sigma = \mathbf{0}$ the construction is consistent with the static representation: f_0 is the constant $v^{-1}(\int u(\underline{f}^\mu, \bar{f}^\mu) d\mu)$, so $V(\mathbf{0}, f_0) = \int u(\underline{f}^\mu, \bar{f}^\mu) d\mu = V(\mathbf{0}, f)$.) Moreover, by **Lemma 1** and (A.2), each interim preference $\succeq_s^\sigma$, $s \in \sigma(\Omega)^+$, is represented by $v^{-1} \circ V_s^\sigma$, hence by V_s^σ . $\square$

A.1.3. Necessity of the axioms

Suppose $\succeq$ is an REUU preference represented, as in **Definition 5**, by the pair $\langle \mu, u \rangle$ of **Assumption 1**, so that $(\sigma, f) \succeq (\tau, g) \iff V(\mathbf{0}, f_\sigma) \geq V(\mathbf{0}, g_\tau)$. Fix a signal σ . Three preliminary observations:

(N1) If two acts $f, \hat{f}$ agree on B_s then $V_s^\sigma(f) = V_s^\sigma(\hat{f})$, since the conditional envelopes are computed from the restriction to B_s alone. More generally, if f and $\hat{f}$ agree except on an event of μ_s^σ -outer measure zero within B_s , their conditional envelopes agree μ_s^σ -almost everywhere and $V_s^\sigma(f) = V_s^\sigma(\hat{f})$; and if $f \geq \hat{f}$ pointwise, then $V_s^\sigma(f) \geq V_s^\sigma(\hat{f})$, since envelopes are ordered almost everywhere and u is weakly increasing in each coordinate.

(N2) For fixed values of the other coordinates, the map assigning to $c \in X$ the value $V(\mathbf{0}, c_{B_s} k)$, where k denotes any σ -adapted act specified off B_s , is weakly increasing by pointwise monotonicity of the static envelopes and weak coordinatewise monotonicity of u , and by **Lemma 2(a)–(b)** it strictly separates $c > c'$ whenever $k \equiv y_0$ with $c' < y_0 < c$ and $B_s \notin \mathcal{N}$.

(N3) For $s \in \sigma(\Omega)^+$ (equivalently $p_s^\sigma > 0$, by **Lemma 2(c)**, whose proof used only **Assumption 1**): the derived relation $\succeq_s^\sigma$ is represented by $v^{-1} \circ V_s^\sigma$. Indeed, writing $c(f) := v^{-1}(V_s^\sigma(f))$, the adapted acts $(f_{B_s} h)_\sigma$ and $(c(f)_{B_s} h)_\sigma$ coincide for every h , so $(\sigma, f_{B_s} h) \sim (\sigma, c(f)_{B_s} h)$; if $c(f) \geq c(g)$ then $(f_{B_s} h)_\sigma \geq (g_{B_s} h)_\sigma$ pointwise for every h , so monotonicity of the static EUU functional gives $f \succeq_s^\sigma g$; if $c(f) < c(g)$, taking $h \equiv y_0 \in (c(f), c(g))$ and applying (N2) refutes $f \succeq_s^\sigma g$.

Axiom 1. If f is adapted to σ then for each $s \in \sigma(\Omega)^+$ the restriction of f to B_s is a constant x_s , whose conditional envelopes are μ_s^σ -almost everywhere equal to x_s ; hence f_σ agrees with f except on the null cells, which by **Lemma 2(b)–(c)** form a μ -null event. Static EUU is invariant to μ -null modifications, so $V(\mathbf{0}, f_\sigma) = V(\mathbf{0}, f) = V(\mathbf{0}, f_0)$, and hence $(\sigma, f) \sim (\mathbf{0}, f)$.

Axiom 2. (ii) If $B \in \mathcal{N}$ then $\mu^*(B) = 0$ by **Lemma 2(b)**, so by **Lemma 2(d)** the event $B \cap B_i$ has μ_i^σ -outer measure zero for every $i \in \sigma(\Omega)^+$. By (N1), f and $g_B f$ share the same conditional certainty equivalent profile, so $f_\sigma = (g_B f)_\sigma$ and the two signal-acts are indifferent. (i) Fix s and f . If $B_s \in \mathcal{N}$, the argument just given shows that altering an act on B_s is a matter of indifference, so both clauses hold for every outcome c . If $s \in \sigma(\Omega)^+$, take $c := v^{-1}(V_s^\sigma(f))$. By (N3), $(\sigma, f_{B_s} h) \sim (\sigma, c_{B_s} h)$ for every $h \in \mathcal{F}$, in particular for every constant continuation, which is the first clause. For the second, let c' satisfy $(\sigma, f_{B_s} y) \sim (\sigma, c'_{B_s} y)$ for every $y \in X$; then $(\sigma, c_{B_s} y) \sim (\sigma, c'_{B_s} y)$ for every y , and a constant y_0 strictly between c and c' would, as in the proof of **Lemma 1**, separate two adapted acts; hence $c' = c$, and $(\sigma, f_{B_s} h) \sim (\sigma, c'_{B_s} h)$ for every h by (N3).

Axiom 3. First, by (N3) and the computation (A.3) of V_s^σ on ideal simple traces (which under the representation is immediate: the conditional envelopes of $x_{E \cap B_s} y$ equal the act itself μ_s^σ -almost everywhere), the identification (A.4) of conditionally null events holds under the representation as well: $A \in \mathcal{N}_s^\sigma$ if and only if $A \cap B_s$ has μ_s^σ -outer measure zero, and interim bounds coincide with μ_s^σ -essential bounds.

(iii) If $A := E \cap B_s \in \mathcal{N}_s^\sigma$ then $f_A g$ and $z_A g$ agree except on a set of μ_s^σ -outer measure zero within B_s and agree off B_s ; by (N1) their conditional certainty equivalent profiles coincide, so the two signal-acts are indifferent.

(i) Suppose the hypothesis of (i) holds for f, h , and $A = E \cap B_s \notin \mathcal{N}_s^\sigma$; write $(x, y) := (\underline{x}(h), \bar{x}(h))$. I claim $\underline{f}^{\mu_s^\sigma} \leq x$ and $\bar{f}^{\mu_s^\sigma} \leq y$ hold μ_s^σ -almost everywhere on A . For the lower envelope: were $\underline{f}^{\mu_s^\sigma} > x + \epsilon$ on a trace $T := E'' \cap A$ with $\mu_s^\sigma(T) > 0$ for some $\epsilon > 0$, then $f \geq \underline{f}^{\mu_s^\sigma} > x + \epsilon$ pointwise on T , so $\{f < x + \epsilon\} \cap T = \emptyset \in \mathcal{N}_s^\sigma$ and $\underline{x}_s^\sigma(f_T) \geq x + \epsilon > \underline{x}(h)$, contradicting the hypothesis at $E' := E''$. For the upper envelope: instantiating the hypothesis at $E' := \Omega$ gives $\bar{x}_s^\sigma(f_A) \leq y$, so for every n the set $\{f > y + 1/n\} \cap A$ lies in $\mathcal{N}_s^\sigma$; their union $\{f > y\} \cap A$ is therefore covered by a trace $\tilde{N} \in \mathcal{E}_s^\sigma$ with $\mu_s^\sigma(\tilde{N}) = 0$, and the pointwise minimum of $\bar{f}^{\mu_s^\sigma}$ with the function equal to y on $A \setminus \tilde{N}$ and to m elsewhere is a measurable majorant of f ; by the minimality of $\bar{f}^{\mu_s^\sigma}$, $\bar{f}^{\mu_s^\sigma} \leq y$ almost everywhere on A . Since u is weakly increasing in each coordinate, $u(\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma}) \leq u(x, y) = V(\mathbf{0}, h)$ almost everywhere on A . Envelope localization across the conditionally ideal partition $\{A, B_s \setminus A\}$ then gives

$$V_s^\sigma(f_A g) = \int_A u(\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma}) d\mu_s^\sigma + C_g \leq \mu_s^\sigma(A) V(\mathbf{0}, h) + C_g,$$

where C_g denotes the (common) contribution of g on $B_s \setminus A$. If $(\mathbf{0}, z) \succeq (\mathbf{0}, h)$ then $v(z) \geq V(\mathbf{0}, h)$, so $V_s^\sigma(z_A g) = \mu_s^\sigma(A) v(z) + C_g \geq V_s^\sigma(f_A g)$. On every other cell $t \neq s$ the two composite acts agree, so by (N1) their conditional certainty equivalents there coincide. On B_s , (N3) and the preceding inequality give the weakly larger certainty equivalent to $z_A g$. Hence $(z_A g)_\sigma \geq (f_A g)_\sigma$ pointwise, and monotonicity of the static EUU functional delivers $(\sigma, z_A g) \succeq (\sigma, f_A g)$.

(ii) is dual: the reversed hypothesis forces $\underline{f}^{\mu_s^\sigma} \geq x$ and $\bar{f}^{\mu_s^\sigma} \geq y$ almost everywhere on A . For the upper envelope: were $\bar{f}^{\mu_s^\sigma} < y - \epsilon$ on a trace T of positive conditional measure, then $f < y - \epsilon$ pointwise on T , so $\bar{x}_s^\sigma(f_T) \leq y - \epsilon < \bar{x}(h)$, contradicting the hypothesis. For the lower envelope: instantiating at $E' := \Omega$ gives $\underline{x}_s^\sigma(f_A) \geq x$, so $\{f < x\} \cap A$ is covered by a μ_s^σ -null trace, and raising the minorant to x off that trace shows, by the maximality of $\underline{f}^{\mu_s^\sigma}$, that $\underline{f}^{\mu_s^\sigma} \geq x$ almost everywhere on A . Hence $u(\underline{f}^{\mu_s^\sigma}, \bar{f}^{\mu_s^\sigma}) \geq u(x, y) = V(\mathbf{0}, h)$ almost everywhere on A , and $(\mathbf{0}, h) \succeq (\mathbf{0}, z)$ yields $V_s^\sigma(\bar{f}_A g) \geq V_s^\sigma(z_A g)$, whence $(\sigma, f_A g) \succeq (\sigma, z_A g)$ as above.

Axiom 4. With g the ideal act taking value x_i on E_i , the conditional envelopes of g on B_s equal g itself μ_s^σ -almost everywhere, so by (N3), $g \succeq_s^\sigma x \iff \sum_i \mu_s^\sigma(E_i \cap B_s) v(x_i) \geq v(x)$. The static expected utility computation in Step 2 shows the right side of the axiom holds if and only if $p_s^\sigma \sum_i \mu_s^\sigma(E_i \cap B_s) v(x_i) \geq p_s^\sigma v(x)$. Since $p_s^\sigma > 0$, the two sides are equivalent. $\square$

Finally, for **Remark 4**: under the representation, restricting $\succeq_s^\sigma$ to acts adapted to $\mathcal{E}_s^\sigma$ on the cell yields the conditional expected utility criterion $f \mapsto \int_{B_s} v \circ f d\mu_s^\sigma$, and both **P2** implications in **Definition 6** (applied to $\succeq_s^\sigma$ and an event $E \cap B_s$) follow from the additive separability of this criterion across $\{E \cap B_s, B_s \setminus (E \cap B_s)\}$ together with (N1) and (N3). The coincidence with (3) for ideal signals was noted in the remark itself.

A.2. Proof of Proposition 1

If $\mu(G) = 1$, the claim follows immediately by taking $\pi = \mu$, so suppose $0 < \mu(G) < 1$ and set $H := \Omega \setminus G$. Write

$$\mu_G(E) := \frac{\mu(E)}{\mu(G)} \quad (E \in \mathcal{E}, E \subseteq G), \quad \mu_H(E) := \frac{\mu(E)}{\mu(H)} \quad (E \in \mathcal{E}, E \subseteq H).$$

Sufficiency. Suppose $\{D_n\}_{n \geq 1}$ is a countably infinite partition of H into events diffuse relative to H . Since μ_G and μ_H are nonatomic, there exist ideal random variables $U_0 : G \rightarrow [0, 1]$ and $U : H \rightarrow [0, 1]$ that are uniform under μ_G and μ_H , respectively. For completeness, each may be constructed by recursively bisecting every cell of a binary ideal partition into two sets of equal probability and reading the resulting binary sequence as a number in $[0, 1]$.

Let $\{q_n\}_{n \geq 1}$ enumerate the rationals in $[0, 1]$ and, with addition understood modulo one, put $U_n := (U + q_n) \bmod 1$. Define $Y : H \rightarrow [0, 1]$ by

$$Y(\omega) := U_n(\omega) \quad \text{if } \omega \in D_n.$$

We first show that, for every Borel $B \subseteq [0, 1]$ with $0 < L(B) < 1$, the event $Y^{-1}(B)$ is diffuse relative to H , where L denotes Lebesgue measure. Fix a non-null ideal event $F \subseteq H$ and define the finite measure

$$\eta_F(C) := \mu_H(F \cap U^{-1}(C)), \quad C \in \mathcal{B}([0, 1]).$$

Since U is uniform, $\eta_F(C) \leq L(C)$, so η_F has a density h_F with respect to L . For $q \in [0, 1]$ set

$$\phi(q) := \eta_F(B - q) = \int_0^1 h_F(t) \mathbf{1}_B(t + q) dL(t),$$

where subtraction and addition are modulo one. Translation is continuous in L^1 , so ϕ is continuous, and Fubini's theorem gives

$$\int_0^1 \phi(q) dL(q) = \mu_H(F) L(B) > 0.$$

Thus $\phi(q) > 0$ on a nonempty open set and hence for some rational q_n . The ideal event $F \cap U_n^{-1}(B)$ is therefore non-null. Since D_n is diffuse relative to H ,

$$F \cap U_n^{-1}(B) \cap D_n \neq \emptyset,$$

so $F \cap Y^{-1}(B) \neq \emptyset$. Applying the same argument to B^c gives $F \cap (H \setminus Y^{-1}(B)) \neq \emptyset$. Hence $Y^{-1}(B)$ is diffuse relative to H .

If $L(B) = 0$, then each $U_n^{-1}(B)$ is an ideal μ_H -null event and

$$Y^{-1}(B) = \bigcup_{n \geq 1} (D_n \cap U_n^{-1}(B))$$

is contained in an ideal μ -null event. By completeness of μ , $Y^{-1}(B)$ is itself ideal and null. The analogous conclusion for $L(B) = 1$ follows by applying this observation to B^c .

Define $Z : \Omega \rightarrow [0, 1]$ by

$$Z(\omega) := \begin{cases} U_0(\omega), & \omega \in G, \\ Y(\omega), & \omega \in H, \end{cases}$$

and let

$$\mathcal{A}_0 := \{Z^{-1}(B) : B \in \mathcal{B}([0, 1])\}.$$

For $A \in \mathcal{A}_0$ define

$$\pi_0(A) := \frac{\mu(A \cap G)}{\mu(G)}.$$

If $A = Z^{-1}(B)$, then $A \cap G = U_0^{-1}(B)$ and hence $\pi_0(A) = L(B)$. It follows directly that π_0 is a countably additive non-atomic probability on the σ -algebra $\mathcal{A}_0$.

Fix $A = Z^{-1}(B) \in \mathcal{A}_0$ and put $p := \pi_0(A)$. If $0 < p < 1$, then $A \cap G$ is an ideal subset of A with

$$\mu(A \cap G) = \mu(G)p,$$

while $A \cap H = Y^{-1}(B)$ is diffuse relative to H . Thus every ideal subset $E \subseteq A$ satisfies $\mu(E \cap H) = 0$, and therefore

$$\mu(E) \leq \mu(A \cap G) = \mu(G)p.$$

Since $A \cap G$ itself is ideal, this proves $\mu_*(A) = \mu(G)\pi_0(A)$. If $p = 0$, the preceding construction makes A an ideal μ -null event. If $p = 1$, A^c is an ideal μ -null event and hence $\mu_*(A) = 1$. Therefore Definition 10 holds on $\mathcal{A}_0$.

Let $(\mathcal{E}^\pi, \pi)$ be the completion of $(\mathcal{A}_0, \pi_0)$. Completion preserves non-atomicity, so $\pi \in \Pi$. Every π_0 -null event is an ideal μ -null event, so each $A \in \mathcal{E}^\pi$ differs from some $A_0 \in \mathcal{A}_0$ by a μ -null ideal event. Consequently,

$$\pi(A) = \pi_0(A_0) = \frac{\mu(A_0 \cap G)}{\mu(G)} = \frac{\mu(A \cap G)}{\mu(G)},$$

and inner measure is unchanged by the same μ -null symmetric difference. Hence the defining neoadditive-source identity also extends to $\mathcal{E}^\pi$. Thus π is the required prior.

Necessity. Conversely, suppose a neoadditive source $\pi \in \Pi$ with the stated identity exists. Since π is non-atomic, construct a π -uniform random variable $W : \Omega \rightarrow [0, 1]$ by the same binary-partition argument used above. Choose a countably infinite Borel partition $\{I_n\}_{n \geq 1}$ of $[0, 1]$ with $0 < L(I_n) < 1$ for every n , and set

$$A_n := W^{-1}(I_n), \quad D_n := A_n \cap H.$$

Then $\{D_n\}_{n \geq 1}$ partitions H . Fix n and write $p_n := \pi(A_n) \in (0, 1)$. By the stated identity, $C_n := A_n \cap G$ is ideal and

$$\mu(C_n) = \mu(G)p_n = \mu_*(A_n).$$

If a non-null ideal event $F \subseteq H$ were contained in D_n , then $C_n \cup F$ would be an ideal subset of A_n with

$$\mu(C_n \cup F) = \mu(G)p_n + \mu(F) > \mu_*(A_n),$$

a contradiction. Hence D_n contains no non-null ideal subset of H . Applying the same argument to A_n^c shows that $H \setminus D_n$ contains no non-null ideal subset either. Thus every D_n is diffuse relative to H , completing the proof. $\square$

References

- Cerreia-Vioglio, S., Corrao, R., Lanzani, G., January 2024. Un-Common preferences, ambiguity, and coordination. Technical Report. Working paper.
- Chateauneuf, A., Eichberger, J., Grant, S., 2007. Choice under uncertainty with the best and worst in mind: neo-additive capacities. *J. Econ. Theory* 137 (1), 538–567.
- Dempster, A.P., 1967. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Stat.* 38, 325–339.
- Denneberg, D., 1994. Conditional (updating) non-additive measures. *Ann. Oper. Res.* 52, 21–42.
- Denneberg, D., 2002. Conditional expectation for monotone measures, the discrete case. *J. Math. Econ.* 37, 105–121.
- Eichberger, J., Grant, S., Kelsey, D., 2007. Updating choquet beliefs. *J. Math. Econ.* 43, 888–899.
- Epstein, L.G., Schneider, M., 2003. Recursive multiple-priors. *J. Econ. Theory* 113, 1–31.
- Epstein, L.G., Zin, S.E., 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica* 57 (4), 937–969.
- Frisch, D., Baron, J., 1988. Ambiguity and rationality. *J. Behav. Decis. Mak.* 1, 149–157.
- Ghirardato, P., 2002. Revisiting savage in a conditional world. *Econ. Theory* 20, 83–92.
- Gilboa, I., Schmeidler, D., 1993. Updating ambiguous beliefs. *J. Econ. Theory* 59, 33–49.
- Grant, S., Kajii, A., Polak, B., 1998. Intrinsic preference for information. *J. Econ. Theory* 83, 233–259.

- Grant, S., Liu, S.L., Yang, J., 2024. A comment on “expected uncertain utility”. *Econometrica* 92 (1), January 247–256.
- Gul, F., Pesendorfer, W., 2014. Expected uncertain utility. *Econometrica* 82 (1), 1–39.
- Gul, F., Pesendorfer, W., 2015. Hurwicz expected utility and subjective sources. *J. Econ. Theory* 159, 465–488.
- Gul, F., Pesendorfer, W., 2021. Evaluating ambiguous random variables from choquet to maxmin expected utility. *J. Econ. Theory* 192, 1–27.
- Hanany, E., Klibanoff, P., 2007. Updating preferences with multiple priors. *Theor. Econ.* 2 (3), 261–298.
- Jaffray, J.-Y., 1992. Bayesian updating and belief functions. *IEEE Trans. Syst. Man Cybern.* 22, 1144–1152.
- Kast, R., Lapied, A., Toquebeuf, P., 2012. Updating choquet capacities: a general framework. *Econ. Bull.* 32 (2), 1495 – 1503.
- Kreps, D.M., Porteus, E.L., 1978. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica* 46 (1), 185–200.
- Lehrer, E., 2005. Updating non-additive probabilities – a geometric approach. *Games Econ. Behav.* 50, 42–57.
- Lyapunov, A., 1940. Sur les fonctions-vecteurs complètement additives. *Bull. Acad. Sci. USSR Math. Ser.* 4, 465 – 478.
- Machina, M.J., 2009. Risk, ambiguity, and the rank-dependence axioms. *Am. Econ. Rev.* 99 (1), 385–392.
- Machina, M.J., Schmeidler, D., 1992. A more robust definition of subjective probability. *Econometrica* 60 (4), 745–780.
- Piermont, E., 2021. Unforeseen evidence. *J. Econ. Theory* 193, 105235.
- Pires, C., 2002. A rule for updating ambiguous beliefs. *Theory Decis.* 53, 137–152.
- Sarin, R.K., Wakker, P.P., 1998. Dynamic choice and non-expected utility. *J. Risk Uncertain.* 17, 87–119.
- Shafer, G., 1976. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, New Jersey.
- Siniscalchi, M., 2011. Dynamic choice under ambiguity. *Theor. Econ.* 6, 379–421.
- Walley, P., 1991. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London.
- Wang, T., 2003. Conditional preferences and updating. *J. Econ. Theory* 108, 286–321.