Comparing three ways to update Choquet beliefs

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A R T I C L E   I N F O

A B S T R A C T

We analyze three rules for updating neo-additive capacities. Only for Generalized Bayesian Updating is relative optimism the same for both updated and unconditional capacities. For updates of the other two, either the updated capacity is fully optimistic or fully pessimistic.

1. SEU and CEU

This paper studies how beliefs can be updated to take account of new information in the context of ambiguity. In subjective expected utility theory (henceforth SEU), choice under uncertainty is perceived as the maximization of the mathematical expectation of the utility function with respect to the subjective probabilities. More precisely, let S denote the set of states and $\mathcal{E}$ the set of events (which for our purposes we shall take to be the set of subsets of S). A simple act is defined to be a function from S to X with finite range, where X denotes the set of outcomes. Let $\mathcal{F}$ denote that set of all simple acts. Then for any SEU maximizer there exists a utility index $u:X \to \mathbb{R}$ and a probability measure π on $\mathcal{E}$, such that her preferences $\succsim$ over $\mathcal{F}$, can be represented by the following functional:

$$U(f) = \sum_u \pi(\{s \in S : u(f(s)) = u\}) \times u. \tag{1}$$

Choquet Expected Utility (henceforth CEU) is the extension of SEU in which the mathematical (Choquet) expectation of the utility function is taken with respect to a capacity rather than a probability. A capacity is any function defined on $\mathcal{E}$, that is normalized ($\nu(\varnothing) = 0$ and $\nu(S) = 1$) and respects set monotonicity (for all E, F in $\mathcal{E}$, $E \subset F \Rightarrow \nu(E) \leq \nu(F)$).[1] Formally, along with the utility index $u(\cdot)$ there

is a capacity $\nu(\cdot)$, such that preferences $\succsim$ over $\mathcal{F}$, can be represented by the Choquet integral:

$$V(f) = \int u(f(s)) d\nu(s) = \sum_u [\nu(\{s \in S : u(f(s)) \geq u\}) - \nu(\{s \in S : u(f(s)) > u\})] \times u. \tag{2}$$

To see the connection with SEU, notice that Eq. (1) can be equivalently expressed as:

$$U(f) = \sum_u [\pi(\{s \in S : u(f(s)) \geq u\}) - \pi(\{s \in S : u(f(s)) > u\})] \times u. \tag{3}$$

Hence, Eq. (2) collapses to Eq. (1) whenever $\nu(\cdot)$ is additive, that is, $\nu(E \cap F) + \nu(E \cup F) = \nu(E) + \nu(F)$, for all E, F in $\mathcal{E}$.

2. Updating SEU

One of the many advantages of SEU is that there is a well-established theory of how preferences should be updated conditional on the knowledge that the state lies in a particular event $E \subset S$. The relationship between the representation of the unconditional preferences and the conditional, can be obtained by updating $\pi(\cdot)$ using the law of conditional probability. More precisely, if $\succsim$ over $\mathcal{F}$ can be represented by a functional of the form given in Eq. (1) then for any event E in $\mathcal{E}$, such that $\pi(E) > 0$, the conditional preferences admit an SEU representation, with the same utility index $u(\cdot)$, and a subjective probability $\pi_E(\cdot)$, given by,

$$\pi_E(A) = \frac{\pi(A \cap E)}{\pi(E)} \text{ for all } A \text{ in } E. \tag{4}$$

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[1] For axiomatizations of CEU see Gilboa (1987), Schmeidler (1989) and Sarin and Wakker (1993).

3. Three ways to update capacities

The situation for CEU is not so well settled. There have been a number of proposals and axiomatizations to characterize the updating of CEU preferences. The three we consider here, all share the feature of SEU that the utility index appearing in the representation of the updated preferences is unchanged. Furthermore, also analogous to updated SEU, it is only the capacity representing the beliefs that is adjusted in response to the information that the state resides in the event E. The three rules are: the optimistic updating rule (see Gilboa and Schmeidler, 1993), the Dempster–Shafer rule (see Dempster, 1968; Shafer, 1976), and the Generalized Bayesian Updating rule (see Dempster, 1967; Fagin and Halpern, 1991; Walley, 1991; Jaffray, 1992).[2] The formal definitions follow.

For any event E in $\mathcal{E}$, let E^c denote the complement of E.

Definition 1. The optimistic updating rule is given by

$$\nu_E^O(A) = \frac{\nu(A \cap E)}{\nu(E)}. \tag{5}$$

Definition 2. The Dempster–Shafer updating rule is given by

$$\nu_E^{DS}(A) = \frac{\nu(A \cup E^c) - \nu(E^c)}{1 - \nu(E^c)}. \tag{6}$$

Definition 3. The Generalized Bayesian Updating rule is given by

$$\nu_E^{GB}(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(A \cup E^c)}. \tag{7}$$

It is straightforward to check that all three reduce to Eq. (4). If the original capacity ν is additive, for Eq. (5) it is immediate. For Eq. (6), notice that additivity implies that $\nu(A \cup E^c) - \nu(E^c) = \nu(A \cap E)$, and $1 - \nu(E^c) = \nu(E)$. For Eq. (7), additivity implies $1 - \nu(A \cup E^c) = \nu(A^c \cap E)$, and applying additivity again yields $\nu(A \cap E) + \nu(A^c \cap E) = \nu(E)$.

To compare and contrast these updating rules, consider the Choquet Expected Utility of a binary bet, or an act of the form 'a bet for x on A and (equivalently) a bet against y on A^c' and that we shall denote by $x_A y$ (where x and y are both in X, with $x \succ y$):

$$V(x_A y) = \nu(A)u(x) + [1 - \nu(A)]u(y). \tag{8}$$

For a given capacity μ, let $\bar{\mu}(\cdot)$ denote the *conjugate* capacity to $\mu(\cdot)$, where for each B in $\mathcal{E}$, $\bar{\mu}(B) := 1 - \mu(B^c)$.[3] With this definition in hand, we see that Eq. (8) can be re-expressed as follows:

$$V(x_A y) = \nu(A)u(x) + \bar{\nu}(A^c)u(y). \tag{9}$$

From Eq. (9) we see that for a simple bet, the decision weight on the favorable event (that is, event A) is given by the capacity of that event, while the decision weight on the unfavorable event (that is, the complement of A) is given by the *conjugate* capacity of that event.

Now let us consider the conditional CEU of a simple bet and how it varies depending on which rule is used to generate the updated capacity (and hence updated conjugate capacity).

If $\nu_E(\cdot)$ denotes the updated capacity conditional on the event E having obtained (and so $\bar{\nu}_E(B) = 1 - \nu_E(B^c)$ is the associated updated conjugate capacity), then the updated CEU of that bet conditional on E having obtained is given by:

$$V_E(x_A y) = \nu_E(A)u(x) + \bar{\nu}_E(A^c)u(y). \tag{10}$$

Applying the optimistic updating rule in Eq. (5) to the right-hand side of Eq. (10), we obtain:

$$V_E^O(x_A y) = \frac{\nu(A \cap E)}{\nu(E)}u(x) + \frac{\nu(E) - \nu(A \cap E)}{\nu(E)}u(y). \tag{11}$$

Applying the Dempster–Shafer updating rule in Eq. (6) to the right-hand side of Eq. (10), yields:

$$\frac{1 - \nu(E^c) - (1 - \nu(A \cup E^c))}{1 - \nu(E^c)}u(x) + \frac{1 - \nu(A \cup E^c)}{1 - \nu(E^c)}u(y).$$

Hence we have,

$$V_E^{DS}(x_A y) = \frac{[\bar{\nu}(E) - \bar{\nu}(A^c \cap E)]}{\bar{\nu}(E)}u(x) + \frac{\bar{\nu}(A^c \cap E)}{\bar{\nu}(E)}u(y). \tag{12}$$

Applying the Generalized Bayesian Updating rule in Eq. (7) to the right-hand side of Eq. (10), yields:

$$\frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(A \cup E^c)}u(x) + \frac{1 - \nu(A \cup E^c)}{\nu(A \cap E) + 1 - \nu(A \cup E^c)}u(y),$$

or

$$V_E^{GB}(x_A y) = \frac{\nu(A \cap E)}{\nu(A \cap E) + \bar{\nu}(A^c \cap E)}u(x) + \frac{\bar{\nu}(A^c \cap E)}{\nu(A \cap E) + \bar{\nu}(A^c \cap E)}u(y). \tag{13}$$

The optimistic updating rule adjusts the decision weight on the *good* outcome according to the law of conditional probability, with the decision weight on the bad outcome determined as the complementary decision weight. The Dempster–Shafer Rule adjusts the decision weight on the *bad* outcome according to the law of conditional probability with the decision weight on the good outcome determined as the complementary decision weight. The Generalized Bayesian Updating rule, on the other hand adjusts both the capacity determining the decision weight on the good outcome and the conjugate capacity determining the weight on the bad outcome in a 'balanced' or 'symmetric' manner which mirrors the Bayesian updating rule for receipt of information. To illustrate this connection, recall that a probability measure (that is, additive capacity) is self-conjugate (that is, $\pi(B) = \bar{\pi}(B)$, for all B in $\mathcal{E}$). This allows us to express the updated expected utility of a simple bet $x_A y$ conditional on the event E obtaining, in the following manner:

$$U_E^B(x_A y) = \frac{\pi(A \cap E)}{\pi(A \cap E) + \bar{\pi}(A^c \cap E)}u(x) + \frac{\bar{\pi}(A^c \cap E)}{\pi(A \cap E) + \bar{\pi}(A^c \cap E)}u(y). \tag{14}$$

4. Updating neo-additive capacities

Perhaps a more compelling argument in favor of the Generalized Bayesian Updating rule can be seen when the capacity that is to be updated exhibits both optimistic and pessimistic attitudes toward uncertainty. A particularly simple and parsimoniously parameterized capacity that exhibits such behavior is the neo-additive capacity introduced by Chateauneuf et al. (2007).

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[2] For the behavioral foundations of the optimistic and Dempster–Shafe updating rules see Gilboa and Schmeidler (1993), and for the Generalized Bayesian Updating rule see Eichberger et al. (2007a) and Horie (2007a). Horie (2007b) characterizes a general updating rule for convex capacities that includes all three rules as special cases. She refers to the Generalized Bayesian Updating rule as the Dempster–Fagin–Halpern (DFH) rule. The term Generalized Bayesian Updating rule is due to Walley (1991).

[3] Notice that any probability measure π (that is, is an additive capacity) is self-conjugate: $\bar{\pi}(E) = 1 - \pi(E^c) = \pi(E)$ for all E in $\mathcal{E}$.

Neo-additive capacities may be viewed as a convex combination of an additive capacity and a special capacity that only distinguishes between whether an event is impossible, possible or certain. To eschew a more detailed discussion about which events are deemed impossible, possible or certain by the individual, we shall restrict our focus to the case where the only null event is the empty set and the only event which is certain is the state space itself or the conditioning event in the case of conditional capacities.

Formally, we define a conditional neo-additive preference relation as follows:

Definition 4. Neo-additive preferences. Fix an event $E \subseteq S$. The capacity ν_E is a conditional neo-additive capacity, if there exists an additive probability π_E with support E such that:

$$\nu_E(A) = \begin{cases} 0 & \text{if } A \cap E = \varnothing, \\ \delta_E \alpha_E + (1-\delta_E)\pi_E(A) & \text{if } A \cap E \notin \{\varnothing, E\}, \\ 1 & \text{if } E \subseteq A, \end{cases}$$

where $\delta_E, \ \alpha_E \in [0,1]$. The relation $\succsim_E$ is a conditional neo-additive preference relation, if there exists a continuous non-constant real-valued function u on X, and a conditional neo-additive capacity ν_E on $\mathcal{E}$ such that for all $f, g \in \mathcal{F}$

$$f \succsim_E g \Longleftrightarrow \int u(f(s))d\nu_E(s) \geqslant \int u(g(s))d\nu_E(s).$$

Hence a conditional neo-additive capacity ν_E is characterized by the tuple $\langle \pi_E, \delta, \alpha \rangle$. The weight $(1-\delta_E)$ on the probability measure π_E may be interpreted as the decision maker's degree of confidence in his "additive beliefs". The remaining weight δ_E may in turn be viewed as the lack of confidence in these beliefs, and depending on the relative degree of optimism (i.e. α_E) a fraction of that "residual" weight is assigned to the best outcome that can obtain in E with the remainder assigned to the worst outcome that can obtain in E.

Straightforward calculation reveals that the Choquet integral of the simple function $u \circ f$ with respect to a neo-additive capacity ν_E is equal to:

$$(1-\delta)E_{\pi_E}[u \circ f] + \delta_E(\alpha_E \cdot \max_{s \in E}\{u(f(s))\} + (1-\alpha_E) \cdot \min_{s \in E}\{u(f(s))\}).$$

The following result shows that the update of a neo-additive capacity under any of these three rules is itself neo-additive. However it is only the Generalized Bayesian Updating rule which leaves optimism unchanged and just updates beliefs and perceived ambiguity. In contrast for the Dempster–Shafer (resp. Optimistic) rule the updated preferences are always ambiguity-averse (resp. ambiguity-loving). Hence with either of these rules, the degree of optimism of the updated capacity differs dramatically from that of the original capacity.

Proposition 1. *Fix a conditioning event $E \subseteq S$, an unconditional neo-additive capacity, ν, characterized by the parameters $\langle \pi, \delta, \alpha \rangle$. Assume $\pi(E) > 0$. Then for all three updating rules the updated capacity, ν_E, is also neo-additive with parameters $\langle \pi_E, \delta_E, \alpha_E \rangle$. In all cases the updated additive probability π_E is the Bayesian update of π,*

$$\pi_E(A) = \frac{\pi(A \cap E)}{\pi(E)}, \text{ for each } A \in \mathcal{E},$$

otherwise π_E is arbitrary. The capacity ν_E satisfies:
1. *for the optimistic updating rule, $\alpha_{E,\alpha}^O = 1, \delta_{E,\alpha}^O = \frac{\delta\alpha}{(1-\delta)\pi(E) + \delta\alpha}$,*

$$\nu_E^O(A) = (1-\delta_{E,\alpha}^O)\pi_E(A) + \delta_{E,\alpha}^O, \text{ if } \varnothing \subsetneqq A \cap E \subsetneqq E;$$

2. *for the pessimistic updating rule, $\alpha_{E,\alpha}^{DS} = 0, \delta_{E,\alpha}^{DS} = \frac{\delta(1-\alpha)}{(1-\delta)\pi(E) + \delta(1-\alpha)}$,*

$$\nu_E^{DS}(A) = (1-\delta_{E,\alpha}^{DS})\pi_E(A), \text{ if } \varnothing \subsetneqq A \cap E \subsetneqq E;$$

3. *for the Generalized Bayesian Updating rule, $\alpha_{E,\alpha}^{GB} = \alpha, \delta_{E,\alpha}^{GB} = \frac{\delta}{(1-\delta)\pi(E) + \delta}$,*

$$\nu_{E,\alpha}^{GB}(A) = \delta_{E,\alpha}^{GB}\alpha + (1-\delta_{E,\alpha}^{GB})\pi_E(A), \text{ if } \varnothing \subsetneqq A \cap E \subsetneqq E;$$

where $\delta_{E,\alpha}^O$ and $\alpha_{E,\alpha}^O$ denote the ambiguity and ambiguity-attitude parameters for the optimistic update of ν, etc.

Proof. From the definition of a neo-additive capacity, we have for any $E \in \mathcal{E}, \nu_E(\varnothing) = 0$ and $\nu_E(E) = 1$. Fix an event A, such that $A \cap E \notin \{\varnothing, E\}$. Applying the expressions (5), (6) and (7) from the definition of the three updating rules we have

$$\begin{aligned} \nu_E^O(A) &= \frac{\nu(A \cap E)}{\nu(E)} \\ &= \frac{(1-\delta)\pi(A \cap E) + \delta\alpha}{(1-\delta)\pi(E) + \delta\alpha} \\ &= \frac{(1-\delta)\pi(E)}{(1-\delta)\pi(E) + \delta\alpha} \times \frac{\pi(A \cap E)}{\pi(E)} + \frac{\delta\alpha}{(1-\delta)\pi(E) + \delta\alpha}; \end{aligned} \quad (15)$$

$$\begin{aligned} \nu_E^{DS}(A) &= \frac{\nu(A \cup E^c) - \nu(E^c)}{1 - \nu(E^c)} \\ &= \frac{(1-\delta)\pi(A \cap E)}{(1-\delta)\pi(E) + \delta(1-\alpha)} \\ &= \frac{(1-\delta)\pi(E)}{(1-\delta)\pi(E) + \delta(1-\alpha)} \times \frac{\pi(A \cap E)}{\pi(E)}; \end{aligned} \quad (16)$$

$$\begin{aligned} \nu_E^{GB}(A) &= \frac{\nu(A \cap E)}{1 - \nu(E^c \cup A) + \nu(A \cap E)} \\ &= \frac{(1-\delta)\pi(A \cap E) + \delta\alpha}{1 - [(1-\delta)\pi(E^c \cup A) + \delta\alpha] + (1-\delta)\pi(A \cap E) + \delta\alpha} \\ &= \frac{(1-\delta)\pi(A \cap E) + \delta\alpha}{(1-\delta)\pi(E) + \delta} \\ &= \frac{(1-\delta)\pi(E)}{(1-\delta)\pi(E) + \delta} \times \frac{\pi(A \cap E)}{\pi(E)} + \frac{\delta}{(1-\delta)\pi(E) + \delta} \times \alpha \end{aligned} \quad (17)$$

$\square$

Remark 1. A similar result applies when $\pi(E) = 0$. Fix an event A, such that $A \cap E \notin \{\varnothing, E\}$. Notice that the definitions of updated capacities are still well-defined provided $\alpha > 0$ and $\delta > 0$. From Eq. (15) we obtain, $\nu_E^O(A) = 1$, from Eq. (16) we obtain $\nu_E^{DS}(A) = 0$, and from Eq. (17) we obtain, $\nu_E^{GB}(A) = \alpha$.

What we find particularly appealing about the Generalized Bayesian update of the neo-additive capacity is that for the conditional preference relation $\succsim_E$, the relative degree of optimism parameter α is the same as it was for the unconditional preference relation. The individual's degree of optimism versus pessimism as embodied in the parameter α is unaffected by conditioning on an event, just as her "risk-attitudes" as embodied in the utility index u remain the same. Her "lack of confidence" parameter δ_E, however, does change and the change is related to the ex ante probability $\pi(E)$ of the conditioning event E. The less likely it was for the conditioning event to arise under her "additive belief" π, the less confidence the individual attaches to the additive component of the updated neo-additive capacity.

Compare this to the updated capacities obtained by applying the optimistic updating rule and the Dempster–Shafer updating rule. With both of these rules, the degree of optimism of the updated capacity differs dramatically from the degree of optimism of the original unconditional capacity. No matter what event $E \subset S$ we condition upon, with the optimistic updating rule, $\alpha_E^O = 1$, and with the Dempster–Shafer updating rule, $\alpha_E^{DS} = 0$. Thus updating changes ambiguity-attitude.

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