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Choice under uncertainty with the best and worst in mind: Neo-additive capacities

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Abstract

We develop the simplest generalization of subjective expected utility that can accommodate both optimistic and pessimistic attitudes towards uncertainty—Choquet expected utility with non-extreme-outcome-additive (*neo-additive*) capacities. A neo-additive capacity can be expressed as the convex combination of a probability and a special capacity, we refer to as a Hurwicz capacity, that only distinguishes between whether an event is impossible, possible or certain. We show that neo-additive capacities can be readily applied in economic problems, and we provide an axiomatization in a framework of purely subjective uncertainty.

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> That the chance of gain is naturally over-valued we may learn from the universal success of lotteries. […] The vain hope of gaining some of the great prizes is the sole cause of this demand. The soberest people scarce look upon it as a folly to pay a small sum for the chance of gaining ten or twenty thousand pounds.
> Adam Smith (1776) “The Wealth of Nations” [28, p. 210]

1. Introduction

Optimism and pessimism are important features of a person’s attitude towards uncertainty. On an aggregate level, business cycles and stock market fluctuations have been attributed to

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'irrational' optimism and pessimism. Economic theory, however, finds it difficult to see in such moods a major factor determining economic behavior. With large amounts of money and wealth at stake, as in the investment behavior of traders in financial markets, one hesitates to attribute major influence on decisions to vague notions of belief.

There are, however, behavioral regularities which influence individuals' betting behavior. People distinguish categorically between situations which they consider as certain, just possible, or strictly impossible. These consistently observed *certainty* and *impossibility effects* cannot be modeled by a transition from zero probability of an event to a positive probability, or from a positive probability to the probability of one. A typical lottery with a high prize on a very unlikely event can turn the certainty of low wealth for a poor person into the possibility of great riches, providing a reason for accepting an unfair gamble. Conversely, rich people may find the possibility of losing substantial amounts of wealth so dangerous that high expected returns are necessary to induce them to an investment. Bell [2] interprets these psychological biases as *disappointment aversion* or *elation-seeking* behavior. Optimistic behavior overestimates the likelihood of good outcomes while pessimistic attitudes exaggerate the likelihood of bad outcomes. Based upon mounting experimental evidence for certainty and impossibility effects, Wakker [30] extends these notions to arbitrary events with rank-ordered outcomes and characterizes *optimistic* and *pessimistic* attitudes. In the context of the Choquet expected utility (CEU) model, concave capacities reflect optimistic attitudes towards uncertainty, while convex capacities model pessimism.

In this paper we propose a weighting scheme which we call a *neo-additive capacity* for reasons explained below. We argue it maintains the key features of the weighting schemes consistently observed in experiments. In economic applications, it allows for an intuitive interpretation of ambiguity and of optimism and pessimism as attitudes towards ambiguity. Most importantly for economic applications, a neo-additive capacity has a simple Choquet integral. Integrating over a lottery yields a weighted average of the expected utility of the lottery and its maximal and minimal outcomes. We also provide a rigorous behavioral foundation for this weighting scheme. [1]

In the context of choice of objective lotteries there are a few studies providing behavioral axioms for functionals which combine expected utility with minimum utility and maximum utility approach. Gilboa [14] and Jaffray [19] independently axiomatize a functional which aggregates the expected utility functional and the minimum utility functional in a monotonic function. Their representations capture only pessimistic attitudes towards uncertainty. Cohen [4], in contrast, provides axioms for a representation of preferences over lotteries as a weighted average of expected utility, minimum utility and maximum utility. For the context of choice over lotteries, her paper provides the behavioral foundations for the representation studied in this paper.

In the next section, we introduce some notation and concepts necessary for our analysis. Section 3 studies the *neo-additive* weighting scheme in the context of the CEU model. This parameterized CEU model can be easily applied to economic models in order to analyze the implications of the certainty and impossibility effect. Section 4 illustrates the potential of the *neo-additive* CEU representation for economic applications in the context of a portfolio choice model. Section 5 provides an axiomatic treatment of neo-additive capacities in a framework of purely subjective uncertainty. Proofs are collected in an appendix.

[1] Ghirardato et al. [10] present a model which generalizes Gilboa and Schmeidler [15] and which also allows for positive and negative attitudes toward ambiguity. Though their framework can also be applied to the representation in this paper, it is irrelevant for the Choquet model with neo-additive capacities as shown in Eichberger et al. [6].

2. Capacities and the choquet integral

We assume that the uncertainty a decision maker faces can be described by a non-empty set of *states*, denoted by S. This set may be finite or infinite. Associated with the set of states is the set of events, taken to be a sigma-algebra of subsets of S, denoted by $\mathcal{E}$. We assume that for each s in S, $\{s\}$ is in $\mathcal{E}$. *Capacities* are real-valued functions defined on $\mathcal{E}$ that generalize the notion of probability distributions. Formally, a capacity is a normalized monotone set function.

Definition 2.1. A *capacity* is a function $v : \mathcal{E} \to \mathbb{R}$ which assigns real numbers to events, such that (i) $E, F \in \mathcal{E}$, $E \subseteq F$ implies $v(E) \leqslant v(F)$ (*monotonicity*), and (ii) $v(\emptyset) = 0$ and $v(S) = 1$ (*normalization*).

A capacity v is called *convex* if $v(E \cup F) \geqslant v(E) + v(F) - v(E \cap F)$ holds for arbitrary events $E, F \in \mathcal{E}$. If the reverse inequality holds then the capacity is called *concave*. Probability distributions are special cases of capacities which are both concave and convex. For each capacity v there is a *dual* or *conjugate capacity* $\overline{v}$ defined by $\overline{v}(E) = 1 - v(S \backslash E)$ for all $E \in \mathcal{E}$. If the dual capacity $\overline{v}$ is *convex*, then the capacity v is *concave*.

The most common way to integrate functions with respect to a capacity is the *Choquet integral*. Let $f : S \to \mathbb{R}$ be a $\mathcal{E}$-measurable real-valued function. We consider finite outcome acts and suppose that f has finite range, that is, the set $f(S)$ is finite. We call a function f with these properties a *simple function*. The Choquet integral can therefore be written in the following intuitive form.

Definition 2.2. For any simple function f the *Choquet integral* with respect to the capacity v is defined as

$$\int f\,dv = \sum_{t \in f(S)} t \cdot \left[v(\{s | f(s) \geqslant t\}) - v(\{s | f(s) > t\}) \right].$$

The Choquet integral is interpreted as the expected value of the function f with respect to the capacity v. The decision weights used in the computation of the Choquet integral will overweight high outcomes if the capacity is concave and will overweight low outcomes if the capacity is convex. It is therefore well-suited to model such responses to ambiguity as *optimism* or *pessimism*. Sarin and Wakker [25] provide a detailed discussion of decision weights.

3. Neo-additive capacities

In this section we introduce a special kind of capacity which we call a *neo-additive capacity* because it is *additive* on *non-extreme outcomes*. Neo-additive capacities may be viewed as a convex combination of an additive capacity and a special capacity that only distinguishes between whether an event is impossible, possible or certain. Since the Choquet integral of an act with respect to this special capacity corresponds to the Hurwicz criterion for decision making under uncertainty, we refer to this non-additive capacity as a *Hurwicz capacity*.

We begin by considering a partition of the set of events into the following three subsets: the set of 'null' events, the set of 'universal' events and the set of 'essential' events, denoted $\mathcal{N}, \mathcal{U}$ and $\mathcal{E}^*$, respectively. As its name suggests, a set is 'null' if 'loosely speaking' it is impossible for it to occur. Formally, we assume that this collection of events satisfies the following properties: (i) $\emptyset \in \mathcal{N}$,

(ii) if $A \in \mathcal{N}$, then $B \in \mathcal{N}$, for all $B \subset A$ and (iii) if $A, B \in \mathcal{N}$ then $A \cup B \in \mathcal{N}$. A 'universal' set is one that is viewed as being certain to occur. Formally, it is the set of events obtained by taking the complements of each member of the set of null events, that is, $\mathcal{U} = \{E \in \mathcal{E} : S \backslash E \in \mathcal{N}\}$. Notice that since $\emptyset \in \mathcal{N}$, it follows from the definition of the set of universal events that $S \in \mathcal{U}$. Furthermore, if $A \in \mathcal{U}$, then from property (ii) for $\mathcal{N}$, it follows that if $A \subset B$ then $B \in \mathcal{U}$. And from property (iii) for $\mathcal{N}$, it follows that if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$. Note that the collection of universal sets $\mathcal{U}$ is a filter as defined in Marinacci [21, p. 1002]. Finally, every other set is 'essential' in the sense that is neither impossible nor certain, that is, $\mathcal{E}^* = \mathcal{E} \backslash (\mathcal{N} \cup \mathcal{U})$.

Definition 3.1. Fix a set of null events $\mathcal{N} \subset \mathcal{E}$. A capacity $v : \mathcal{E} \to [0, 1]$ is *congruent* with $\mathcal{N}$ if $v(E) = 0$ and $v(S \backslash E) = 1$, for all $E \in \mathcal{N}$. Furthermore the capacity is *exactly congruent* if $v(E) > 0$, for all $E \notin \mathcal{N}$.

For a given set of null events, the Hurwicz capacity is a minimally discriminating capacity that is exactly congruent with the set of null events. That is, it assigns only one of three possible values to an event depending on whether the event is 'impossible' (that is, null), 'possible' (that is, essential) or 'certain' (that is, universal).

Definition 3.2. Fix the set of null events $\mathcal{N} \subset \mathcal{E}$ and fix $\alpha \in [0, 1]$. The Hurwicz capacity exactly congruent with $\mathcal{N}$ and with an α degree of optimism is defined to be

$$\mu_\alpha^{\mathcal{N}}(E) = \begin{cases} 0 & \text{if } E \in \mathcal{N}, \\ \alpha & \text{if } E \notin \mathcal{N} \text{ and } S \backslash E \notin \mathcal{N}, \\ 1 & \text{if } S \backslash E \in \mathcal{N}. \end{cases}$$

The Hurwicz capacity may be viewed as a convex combination of two capacities, one of which reflects *complete ignorance* or *complete ambiguity* in everything bar a universal event *occurring*, and the second which reflects *complete confidence* in everything bar null events. Given the filter $\mathcal{U}$, define $\mu^{\mathcal{U}}(E) = 1$ if $E \in \mathcal{U}$ and $\mu^{\mathcal{U}}(E) = 0$ otherwise, and define $\mu^{\mathcal{N}}(E) = 0$ if $E \in \mathcal{N}$ and $\mu^{\mathcal{N}}(E) = 1$ otherwise. By construction, we have $\mu^{\mathcal{U}} = \bar{\mu}^{\mathcal{N}}$, and $\mu_\alpha^{\mathcal{N}}(E) = \alpha\mu^{\mathcal{N}} + (1 - \alpha)\bar{\mu}^{\mathcal{N}}$.

Formally, we define a neo-additive capacity as a convex combination of a Hurwicz capacity and a congruent additive capacity.

Definition 3.3. For a given set of null events $\mathcal{N} \subset \mathcal{E}$, a finitely additive probability distribution π on $(S, \mathcal{E})$ that is congruent with $\mathcal{N}$ and a pair of numbers $\delta, \alpha \in [0, 1]$, a neo-additive capacity $v(.|\mathcal{N}, \pi, \delta, \alpha,)$ is defined as

$$v(E|\mathcal{N}, \pi, \delta, \alpha) := (1 - \delta)\pi(E) + \delta\mu_\alpha^{\mathcal{N}}(E).$$

It is straightforward to derive the Choquet integral of a simple function f with respect to a neo-additive capacity.

Lemma 3.1. *The Choquet expected value of a simple function* $f : S \to \mathbb{R}$ *with respect to the neo-additive capacity* $v(\cdot|\mathcal{N}, \pi, \delta, \alpha)$ *is given by*

$$\int f\,dv := (1 - \delta)E_\pi[f] + \delta\left(\alpha \cdot \max\left\{x : f^{-1}(x) \notin \mathcal{N}\right\} + (1 - \alpha) \cdot \min\left\{y : f^{-1}(y) \notin \mathcal{N}\right\}\right).$$

Proof. To see this note that $V(f|\mu_\alpha^{\mathcal{N}}(\cdot)) = \alpha \cdot \max\left\{x : f^{-1}(x) \notin \mathcal{N}\right\} + (1-\alpha) \cdot \min\left\{y : f^{-1}(y) \notin \mathcal{N}\right\}$ and $V(f|\pi) = E_\pi[f]$. The result then follows from the linearity of the Choquet integral with respect to the capacity (Denneberg [5], Properties (ix) and (x) on p. 49). □

Several well-known decision criteria can be viewed as special cases of the Choquet integral of a neo-additive capacity:

(i)	$\delta = 0$	*expected utility*,
(ii)	$\mathcal{N} = \{\emptyset\}, \delta > 0, \alpha = 0,$	*pure pessimism*,
(iii)	$\mathcal{N} = \{\emptyset\}, \delta > 0, \alpha = 1$	*pure optimism*,
(iv)	$\mathcal{N} = \{\emptyset\}, \delta = 1, \alpha \in (0, 1)$	*Hurwicz criterion*.

Neo-additive capacities satisfy three conditions:

- They are additive for pairs of disjoint events which are not null and do not form a partition of a universal event.
- They exhibit uncertainty aversion for some events.
- They exhibit uncertainty preference for some other events.

A decision maker with beliefs represented by neo-additive capacities shows on some events optimistic and on others pessimistic behavior in the sense of Wakker [30]. Moreover, it is straightforward to check that of two decision makers with the same beliefs and degrees of ambiguity (π, δ), the one with the smaller α is more ambiguity-averse in the sense of Epstein [8] and Ghirardato and Marinacci [13].

Indeed, as the following proposition shows, these conditions characterize neo-additive capacities completely.

Proposition 3.1. *Fix the set of null events $\mathcal{N} \subset \mathcal{E}$, so that $\mathcal{E}^*$ contains at least three elements E_1, E_2 and E_3 which are pairwise disjoint (that is, $E_i \cap E_j = \emptyset$ for all $i \neq j$). Let v be a capacity on $(S, \mathcal{E})$ congruent with $\mathcal{N}$. Then the following statements are equivalent*:

(i) *v is a neo-additive capacity*,

(ii) *the capacity v satisfies the following properties*:

  (a) *for any three events $E, F, G \in \mathcal{E}^*$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \in \mathcal{E}^*$, and $E \cup G \in \mathcal{E}^*$, $v(E \cup F) - v(F) = v(E \cup G) - v(G)$*,

  (b) *for some $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \in \mathcal{E}^*$, $v(E \cup F) \leqslant v(E) + v(F)$*,

  (c) *for some $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \in \mathcal{E}^*$, $\overline{v}(E \cup F) \leqslant \overline{v}(E) + \overline{v}(F)$*,

  (d) *for any $E \in \mathcal{E}^*$ and any $F \in \mathcal{N}$ such that $E \cap F = \emptyset$, $v(E \cup F) = v(E)$.*

Proof. Appendix.

Note that for a neo-additive capacity v on $(S, \mathcal{E})$, where $\mathcal{E}$ contains singletons and at least three essential events, as assumed throughout the paper, uniqueness of the pessimism and optimism coefficients and of the underlying probability measure π is guaranteed. This is proved in a lemma preceding the proof of Proposition 3.1 in the appendix.

Property (iia) establishes additivity of the neo-additive capacity for events that yield non-extreme outcomes. According to property (iib), the capacity (weakly) overweights the event in which the most preferred prize is obtained, hence $\alpha \geqslant 0$. Property (iic) says the capacity also (weakly) overweights the event with the least preferred prize. It implies $\alpha \leqslant 1$. Property (iid) finally is trivially satisfied if $\mathcal{N} = \{\emptyset\}$.

Remark 3.1 (*Probabilistic sophistication*). If a neo-additive capacity $v(\cdot|\mathcal{N}, \pi, \delta, \alpha)$ is absolutely and mutually continuous with respect to its 'additive' component π [that is, for any event E, $\pi(E) = 0$ implies $E \in \mathcal{N}$ (and hence, $v(E) = 0$)], then the preferences over acts generated by the Choquet integral with respect to that capacity are probabilistically sophisticated with respect to π in the sense of Machina and Schmeidler [20]. That is, we may view the choice behavior rationalized by these preferences as being based on probabilistic beliefs, represented by the probability measure π.

To see this, note that by using π, any act f can be mapped to a simple lottery over outcomes $P = \pi \circ f^{-1}$. If we take $W(\cdot)$ to be the mapping from the set of simple probability distributions on $\mathbb{R}$, to $\mathbb{R}$, given by

$$\begin{aligned} W(P) = (1-\delta) \sum_{x \in \text{supp}\, P} p(x)\, x + \delta\, (\alpha \cdot \max\{x : x \in \text{supp}\, P\} \\ + (1-\alpha) \cdot \min\{y : y \in \text{supp}\, P\}), \end{aligned}$$

where supp P is the support of P, then from Lemma 3.1 we have $\int f\, dv = W\left(\pi \circ f^{-1}\right)$ for all simple functions f. Hence the preferences over simple lotteries generated by $W(\cdot)$ in conjunction with the probability measure π, fully characterize the underlying preference over acts.

Remark 3.2 (*Multiple priors*). It is well-known that the CEU approach is equivalent to the multiple prior approach if capacities are convex or if they are concave. [2] In general, however, neo-additive capacities are neither convex nor concave. But by simple manipulation of Definition 3.3 we see that a neo-additive capacity may be expressed as

$$\begin{aligned} v &= \alpha \left[(1-\delta)\,\pi + \delta \mu_1^{\mathcal{N}}\right] + (1-\alpha)\left[(1-\delta)\,\pi + \delta \bar{\mu}_1^{\mathcal{N}}\right] \\ &= \alpha\rho + (1-\alpha)\,\bar{\rho}, \end{aligned}$$

where $\rho = (1-\delta)\,\pi + \delta\mu_1^{\mathcal{N}}$ and $\bar{\rho}$ is its dual. Since ρ is concave (and, hence $\bar{\rho}$ is convex), by applying well-known results, we obtain

$$\int f\, dv = \alpha \max_{\hat{\pi} \in \mathcal{D}} \int f\, d\hat{\pi} + (1-\alpha) \min_{\hat{\pi} \in \mathcal{D}} \int f\, d\hat{\pi}, \tag{1}$$

where $\mathcal{D}$ is the core of $\bar{\rho}$, that is,

$$\mathcal{D} = \left\{\hat{\pi} : \hat{\pi}(E) = 0, \text{ for all } E \in \mathcal{N}, \text{ and } \hat{\pi}(E) \geqslant (1-\delta)\,\pi(E), \text{ for all } E \notin \mathcal{N}\right\}.$$

This 'multiple-prior' representation of a neo-additive capacity suggests a separation between ambiguity, reflected by the set $\mathcal{D}$, and attitudes towards ambiguity, reflected by the degrees of optimism α and pessimism $(1-\alpha)$. [3]

4. Economic applications

The ups and downs of economic activity during the business cycle which are usually accompanied by swings in investors' sentiments, ranging from bull to bear spirits in financial markets, provide numerous examples of the impact of uncertainty on economic behavior.

[2] See for instance Gilboa and Schmeidler [16, Theorem 2.2] and its references.

[3] We thank a referee for suggesting a succint way to present this multiple priors representation of the Choquet integral with respect to a neo-additive capacity.

Neo-additive capacities provide a natural way for modeling optimism and pessimism influencing economic activities. The parameters of a neo-additive capacity can be interpreted as measuring confidence in beliefs and degrees of optimism and pessimism. A neo-additive capacity $v(E|\pi, \delta, \alpha)$ is based on an additive probability distribution π reflecting the subjective beliefs of the decision maker. It represents an assessment of the likelihood of events consistent with the individual's belief. The weight $(1 - \delta)$ given to π is a measure of the *degree of confidence* which the individual holds in this belief. The core belief of a neo-additive capacity represented by the additive probability distribution π can be determined endogenously in equilibrium.[4] Thus, standard equilibrium analysis is always the special case of full confidence, $\delta = 0$. Positive parameters $\delta\alpha$ and $\delta(1 - \alpha)$ represent the impact of pessimism and optimism, respectively. Neo-additive capacities can therefore model psychological phenomena such as excessive optimism and pessimism which have been put forward as explanations for economic behavior in depressions or bubbles and which have been confirmed in laboratory experiments.

In this section we show by example that optimism and pessimism can explain behavior inconsistent with expected utility maximization. In these cases, optimism and pessimism can help to explain well-known economic puzzles. We will reconsider the paradox of people buying insurance and gambling, and we will review portfolio choice behavior where one observes unreasonably high risk premia (the *equity premium puzzle*) and a willingness to invest in high-risk stock of unknown start-up companies (the *small stock puzzle*).

4.1. Insurance and gambling

The same individual is often observed to buy both insurance against risk and lottery tickets. As our introductory quotation of Adam Smith illustrates, such behavior is ubiquitous but hard to reconcile with rational decision making based on probabilistic calculus. For expected utility maximizers with a von Neumann–Morgenstern utility function such behavior is hard to explain.[5] Buying insurance suggests a preference for reduced risk, while paying for a lottery implies preference for a risky gamble, often at very unfair odds.

To see how both types of behavior can be accommodated by a neo-additive capacity, consider an individual endowed with wealth x, whose preferences over lotteries can be represented by the CEU of a neo-additive capacity, with parameters $\delta > 0$ and $\alpha < \frac{1}{2}$ for the neo-additive capacity and utility index u (taken to be concave). This individual faces a (small) probability π_L of incurring a loss of size L. Insurance coverage is available at a premium q. Also available at a price p is a lottery ticket that 'wins' with (a very small) probability π_W and pays out the single prize of size W and otherwise pays out nothing. Suppose that the individual views the event in which he incurs the loss and the event in which he wins the lottery (should he purchase a ticket) are independent. Further suppose that $\left[\delta(1 - \alpha) + (1 - \delta) \cdot \pi_L\right] \cdot L > q \geqslant \pi_L \cdot L$. The weak inequality is a feasibility condition for the insurance premium to cover at least the expected loss (and if strict it means that the insurance coverage is actuarially unfair). The strict inequality is satisfied if the individual has a positive degree of pessimism (i.e. $\alpha < 1$) and if the potential loss L is sufficiently large.

[4] Eichberger and Kelsey [7] provide a thorough analysis of strategic games when beliefs are modeled as non-additive capacities.

[5] Friedman and Savage [9] suggest an S-shaped von Neumann–Morgenstern utility function. This approach to reconcile such behavior has been criticized by Markowitz [22]. See Hirshleifer and Riley [18, pp. 26–28] for a discussion of the Friedman–Savage approach.

The difference in the CEUs between buying and not buying insurance is

$$\begin{aligned}
&u(x-q)-\left(\left[\delta\alpha+(1-\delta)\cdot(1-\pi_L)\right]\cdot u(x)+\left[\delta(1-\alpha)+(1-\delta)\cdot\pi_L\right]\cdot u(x-L)\right)\\
&\quad\geqslant u(x-q)-u\left(\left[\delta\alpha+(1-\delta)\cdot(1-\pi_L)\right]\cdot x\right.\\
&\qquad\left.+\left[\delta(1-\alpha)+(1-\delta)\cdot\pi_L\right]\cdot(x-L)\right)\\
&\quad=u(x-q)-u\left(x-\left[\delta(1-\alpha)+(1-\delta)\pi_L\right]L\right)>0.
\end{aligned}$$

The first inequality follows from Jensen's inequality applied to the convex function $-u$, and the second inequality follows from monotonicity of u.

Having purchased the insurance, the difference in CEUs between buying the lottery ticket and not may now be expressed as

$$\begin{aligned}
&\left[\delta\alpha+(1-\delta)\cdot\pi_W\right]\cdot u(x+W-p-q)\\
&\quad+\left[\delta(1-\alpha)+(1-\delta)\cdot(1-\pi_W)\right]\cdot u(x-p-q)-u(x-q)\\
&\qquad>\delta\alpha\cdot u(x+W-p-q)+(1-\delta\alpha)\cdot u(x-p-q)-u(x-q)\\
&\qquad\geqslant\delta\alpha\cdot[u(x+W-p-q)-u(x-q)]-p\cdot u'(x-q).
\end{aligned}$$

The last inequality follows from the concavity of u. For $\delta\alpha>0$ and u strictly increasing, there is a lottery (W,π_W,p) with W high enough and π_W small enough such that $\pi_W\cdot W<p$ and

$$\frac{\delta\alpha\cdot[u(x+W-p-q)-u(x-q)]}{p}>u'(x-q).$$

Notice that this is true for any degree of concavity of u. Optimism makes lotteries with high prizes and low probabilities of winning attractive even for individuals who are averse to accepting actuarially fair 50–50 gambles.

4.2. *Portfolio choice*

There are numerous puzzles in portfolio choice theory. Thaler [29] provides a stimulating exposition of some well-known irregularities. These puzzles highlight inconsistencies between standard economic theories and empirical regularities. Naturally, not all can be related to optimism or pessimism. The following two puzzles, however, can be explained easily by a small degree of optimism and pessimism.

The *equity premium puzzle* refers to the large difference between the average return on a stock portfolio and the return of a fixed interest bearing bond which was first noted by Mehra and Prescott [23]. The implied risk premium appears to be too big to be explained by risk aversion as modeled by a concave von Neumann–Morgenstern utility function. The conservative behavior in the face of uncertainty suggested by such a high risk premium stands in stark contrast to the observation that small firms with high-risk stocks seem to attract investors' interest more than is warranted by their average returns. To invest in stock of young 'promising' companies appears to be extraordinarily risky. Yet such uncertainty did not deter investors who otherwise requested a surprisingly high risk premium.

We study a simple financial market system with a representative investor, one risky and one riskless asset and an exogenous supply of assets. This framework suffices to illustrate the impact of optimism and pessimism on portfolio choice. With well-known modifications these results carry over to more general models of financial markets.

Consider an investor with initial wealth W_0 who can invest in two assets, a stock with uncertain returns and a bond with a certain payoff. The following table summarizes the notation of the assets:

Asset	Quantity	Price	Payoff in state $s \in S$
Stock	a	q	r_s
Bond	b	1	r

Preferences of the investor are represented by a CEU $V(W_1, \ldots, W_S)$ of end-of-period wealth, $W_s = r_s \cdot a + r \cdot b$, with respect to a neo-additive capacity

$$\begin{aligned} &V(W_1, \ldots, W_S) \\ &\quad := \delta\left[\alpha \cdot \max\{u(W_1), \ldots, u(W_S)\} + (1-\alpha) \cdot \min\{u(W_1), \ldots, u(W_S)\}\right] \\ &\qquad + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u(W_s). \end{aligned}$$

Using the budget constraint, $W_0 = q \cdot a + b$, to substitute for the bond, one gets wealth as a function of stock transactions a,

$$W_s = r \cdot W_0 + [r_s - q \cdot r] \cdot a.$$

Denoting by $\overline{r} = \max\{r_1, \ldots, r_S\}$ and $\underline{r} = \min\{r_1, \ldots, r_S\}$ the maximal and minimal returns of the risky stock, one can write the CEU from a stock investment $a > 0$ as

$$\begin{aligned} V(a) := {} & \delta\alpha \cdot u(r \cdot W_0 + [\overline{r} - q \cdot r] \cdot a) + \delta(1-\alpha) \cdot u(r \cdot W_0 + [\underline{r} - q \cdot r] \cdot a) \\ & + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u(r \cdot W_0 + [r_s - q \cdot r] \cdot a). \end{aligned}$$

For a stock market equilibrium price q^* with an aggregate endowment of equity $A > 0$ and bonds $B = 0$ where the single investor maximizes CEU $V(a)$,

$$\begin{aligned} V'(A) = {} & \delta\alpha \cdot u'(r \cdot W_0 + [\overline{r} - q^* \cdot r] \cdot A) \cdot [\overline{r} - q^* \cdot r] \\ & + \delta(1-\alpha) \cdot u'(r \cdot W_0 + [\underline{r} - q^* \cdot r] \cdot A) \cdot [\underline{r} - q^* \cdot r] \\ & + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r \cdot W_0 + [r_s - q^* \cdot r] \cdot A) \cdot [r_s - q^* \cdot r] = 0 \end{aligned}$$

must hold in equilibrium. Substituting for the initial wealth $W_0 = q^* \cdot A$, this equilibrium condition can be solved explicitly for the equilibrium stock price q^*,

$$q^* = \frac{\delta\left[\alpha \cdot u'(\overline{r} \cdot A) \cdot \overline{r} + (1-\alpha) \cdot u'(\underline{r} \cdot A) \cdot \underline{r}\right] + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \left[\delta\left[\alpha \cdot u'(\overline{r} \cdot A) + (1-\alpha) \cdot u'(\underline{r} \cdot A)\right] + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}. \tag{2}$$

The case of subjective expected utility, $\delta = 0$, is the reference situation against which we can assess the impact of optimism or pessimism. Denote by q_0^* the equity price in this case,

$$q_0^* = \frac{\sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)}.$$

The *equity premium* is defined as the ratio

$$\rho(q^*) := \frac{\sum_{s \in S} \pi_s \cdot r_s}{q^* \cdot r}.$$

The smaller q^* the greater the equity premium.

Consider first the case of a risk-neutral investor, $u'(\cdot) = k$. In this case, the equilibrium stock price equals the discounted expected return of the stock plus an optimism and pessimism related premium

$$q^* = \frac{\sum_{s\in S} \pi_s \cdot r_s}{r} + \frac{\delta\alpha \cdot \left(\overline{r} - \sum_{s\in S} \pi_s \cdot r_s\right) + \delta\,(1-\alpha) \cdot \left(\underline{r} - \sum_{s\in S} \pi_s \cdot r_s\right)}{r}$$
$$= q_0^* + \frac{\delta}{r} \cdot \left[\alpha \cdot (\overline{r} - \mathcal{E}_\pi r_s) + (1-\alpha) \cdot \left(\underline{r} - \mathcal{E}_\pi r_s\right)\right],$$

where we denote the expected return of the stock by $\mathcal{E}_\pi r_s := \sum_{s\in S} \pi_s \cdot r_s$.

Since $\overline{r} > \mathcal{E}_\pi r_s > \underline{r}$, optimism will add a positive premium, $\delta\alpha \cdot (\overline{r} - \mathcal{E}_\pi r_s)$, to the reference price q_0^*, while pessimism will make the premium negative, $\delta(1-\alpha) \cdot \left(\underline{r} - \mathcal{E}_\pi r_s\right)$. For a risk-neutral investor, we can note that:

- the equity premium will be the higher the more pessimistic the investor is, i.e., the smaller is α;
- if there is no optimism, $\alpha = 0$, then the equity premium required by a pessimistic investor will be strictly higher than the one based on the subjective probability distribution π alone, $\rho(q^*) > \rho(q_0^*)$;
- if both optimism and pessimism are present, but $\alpha < \frac{1}{2}$ as in most experimental studies, then a sufficient condition for an equity premium $\rho(q^*)$ exceeding $\rho(q_0^*)$ is an average return exceeding the average of the minimum and the maximum return,

$$\frac{\overline{r} + \underline{r}}{2} < \mathcal{E}_\pi r_s.$$

For a risk-averse investor with a strictly decreasing marginal utility function $u'(\cdot)$ similar results can be deduced which we summarize in a lemma.

Lemma 4.1. *A* risk-averse *and* pessimistic *investor, i.e., with* $\alpha = 0, \delta > 0, u'(\cdot)$ *strictly decreasing, requires an equity premium exceeding the equity premium with risk but no pessimism,*

$$\rho(q^*) > \rho(q_0^*).$$

Proof. Appendix.

In recent years 'new stock markets' have emerged in many developed countries where stock of start-up firms is traded. These markets were opened in order to provide venture capital for new high-risk enterprises with great potential. In the light of the rather conservative behavior reflected in the equity premium puzzle it is even more surprising that investors were willing to bet substantial amounts of wealth on firms with no record of earnings.

Optimism and pessimism as modeled with a neo-additive capacity enables us to explain such behavior. In fact, we can show that for an arbitrary small degree of optimism there are maximal returns of a firm high enough to induce a positive stock price for high-risk firms with potentially high returns. Reconsider the stock market equilibrium price of Eq. (2) and assume, without loss of generality, that the firm's stock pays off a return R only in state 1. Hence, $\overline{r} = r_1 = R$ and $\underline{r} = r_s = 0$ for all $s \neq 1$. Suppose the expected return of the firm is bounded away from zero, $\pi_1 \cdot R \geqslant \kappa > 0$. Then the equilibrium price satisfies

$$
\begin{aligned}
q^* &= \frac{1}{r} \cdot \frac{\delta\alpha \cdot u'(R \cdot A) \cdot R + (1-\delta) \cdot \pi_1 \cdot u'(R \cdot A) \cdot R}{\delta\left[\alpha \cdot u'(R \cdot A) + (1-\alpha) \cdot u'(0)\right] + (1-\delta) \cdot \left[\pi_1 \cdot u'(R \cdot A) + \sum_{s \neq 1} \pi_s \cdot u'(0)\right]} \\
&= \frac{R}{r} \cdot \frac{u'(R \cdot A) \cdot \left[\delta\alpha + (1-\delta) \cdot \pi_1\right]}{u'(R \cdot A) \cdot \left[\delta\alpha + (1-\delta) \cdot \pi_1\right] + u'(0) \cdot \left[\delta\,(1-\alpha) \cdot + (1-\delta) \cdot (1-\pi_1)\right]} \\
&\geqslant \frac{R}{r} \cdot \left[\delta\alpha + (1-\delta) \cdot \pi_1\right] \\
&> \frac{R}{r} \cdot \delta\alpha,
\end{aligned}
$$

where the first inequality follows from $u'(0) \geqslant u'(R \cdot A)$ and the second strict inequality from the positive expected return.

It is clear that with some optimism, $\delta\alpha > 0$, even a vanishing probability of success π_1 will not deter investors provided the return rises sufficiently, $R \geqslant \kappa/\pi_1$. The stock market price will not collapse. There is no contradiction if investors buy high-risk stock because of optimism, $\delta\alpha > 0$, and require an 'excessive' equity premium. Adam Smith's [28, p. 210] observation that even 'sober people' do play lotteries and Robert Shiller's [27] observed 'exuberance' in the stock market can be reconciled with rational decision making under uncertainty, if one allows for optimism and pessimism as modeled by neo-additive capacities.

5. Behavioral axioms

We present our theory in the context of a variant of Savage's [26] purely subjective uncertainty framework employed by Ghirardato and Marinacci [12] and Ghirardato, Maccheroni, Marinacci and Siniscalchi [11] (hereafter, GMMS). The state space S is taken to be the same as was defined in Section 2. Let X, the set of outcomes, be a connected and separable topological space. An *act* is a function (measurable with respect to $\mathcal{E}$) $f : S \to X$ with *finite* range. $\mathcal{F}$ denotes the set of such acts and is endowed with the product topology induced by the topology on X. We shall identify each $x \in X$ with the constant act, $f(s) = x$ for all $s \in S$. For any pair of acts f, g in $\mathcal{F}$ and any event $E \in \mathcal{E}$, $f_E g$ will denote the act $h \in \mathcal{F}$, formed from the concatenation of the two acts f and g, in which $h(s)$ equals $f(s)$ if $s \in E$, and equals $g(s)$ if $s \notin E$.

Let $\succsim$ denote the individual's preference relation on $\mathcal{F}$. For any $f \in \mathcal{F}$, the *certainty equivalent* of f, denoted by $m(f)$, is the set of constant acts that are indifferent to f. That is, $x \in m(f)$, if $x \sim f$. Although many constant acts may be equivalent, when there is no risk of confusion, we shall write $m(f)$ to indicate an arbitrary member of the set.

We say f and g are comonotonic if for every pair of states s and s' in S, $f(s) \succ f(s')$ implies $g(s) \succsim g(s')$. An event $E \in \mathcal{E}$ is *null* (*universal*) if $x_E y \sim y$ ($x_E y \sim x$) for all pairs of outcomes $x, y \in \mathcal{X}$ with $x \succ y$. An event E is *essential* if for some $x, y \in X$, $x \succ x_E y \succ y$. Let $\mathcal{N}, \mathcal{U}$ and $\mathcal{E}^*$ denote the sets of null, universal and essential events, respectively. [6]

Neo-additive capacities are a special case of the CEU theory. In order to obtain a behavioral characterization, we seek to modify the axioms of GMMS [11] appropriately. Their key innovation is to define a behavioral definition of 'subjective mixtures' of acts which allows them to define

[6] The definition of null, universal and essential events follows Ghirardato and Marinacci [12]. Together with Axiom 6, null and universal events are also null and universal in the sense of Savage [26].

in a Savage framework of purely subjective uncertainty, analogs to axioms based on probability mixtures that play such a key role in the Anscombe–Aumann [1] framework.

The first axiom restricts preferences to non-trivial cases.

Axiom 0 (*Non-trivial preferences*).

(i) There exist outcomes 0 and M in X, such that $M \succ 0$ and $M \succsim x \succsim 0$ for all $x \in X$.
(ii) There exists $E \in \mathcal{E}$ such that $M \succ M_E 0 \succ 0$.

The first part of Axiom 0 assumes that a *best* and a *worst* outcome exists. The second part rules out that all events are inessential. We will call a preference relation $\succsim$ on $\mathcal{F}$ *non-trivial* if it satisfies Axiom 0.

The next is the standard ordering axiom.

Axiom 1 (*Ordering*).

(i) The preference relation $\succsim$ on $\mathcal{F}$ is complete, reflexive and transitive.

The neo-additive expected utility representation allows for the 'discontinuous over-weighting' of events on which extreme, i.e., either best or the worst, outcomes obtain. Hence, standard continuity with respect to the product topology cannot be expected to hold for the whole preference relation. Following Ghirardato and Marinacci [12] we only require a weaker notion of pointwise convergence, where in this product topology, we say a net $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$ converges pointwise to $f \in \mathcal{F}$, if and only if $f_\alpha(s) \to f(s)$ for all $s \in S$.

Axiom 2 (*Continuity*). Let $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$ be a net that converges pointwise to f and such that all f_αs and f are measurable with respect to the *same* finite partition. If $f_\alpha \succsim g$ (respectively, $g \succsim f_\alpha$) for all $\alpha \in D$, then $f \succsim g$ (respectively, $g \succsim f$).

We also adopt the monotonicity axiom of Chew and Karni [3] which combines statewise dominance with a weakening of Savage's axiom P3.

Axiom 3 (*Eventwise monotonicity*).

(i) For any pair of acts, $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.
(ii) In addition, for any triple of outcomes $x, y, z \in X$, and any event E,
   (a) if $x \succsim z$, $y \succsim z$ then $x \succ y \Rightarrow x_E z \succ y_E z$ for every event $E \notin \mathcal{N}$;
   (b) if $z \succsim x$, $z \succsim y$ then $x \succ y \Rightarrow z_E x \succ z_E y$ for every event $E \notin \mathcal{U}$

The next axiom due to Ghirardato and Marinacci [12] builds on the idea of Nakamura [24] and Gul [17] of a 'subjective mixture' of two acts f and g. Fix some event E, and then construct state by state an act which yields at each state s, the certainty equivalent of the bet $f(s)_E\, g(s)$. Formally, the statewise (event) E-mixture of f and g, denoted as $f^E g$, is taken to be the act

$$f^E g(s) = m\left(f(s)_E\, g(s)\right).$$

Adopting the shorthand $\{x, y\} \succsim z$ for $x \succsim z$ and $y \succsim z$, and $z \succsim \{x, y\}$ for $z \succsim x$ and $z \succsim x$, the next axiom may be stated as follows.

Axiom 4 (*Binary comonotonic act independence*). Consider events $A, B \in \mathcal{E}$, where A satisfies $M \succ M_A 0 \succ 0$, and the acts $f := x_A y$, $g := x'_A y'$, $h := x''_A y''$. If f, g, h are pairwise comonotonic, and $\{x, x'\} \succsim x''$ and $\{y, y'\} \succsim y''$ (or $x'' \succsim \{x, x'\}$ and $y'' \succsim \{y, y'\}$), then

$$f \succsim g \Rightarrow f^B h \succsim g^B h.$$

As its name suggests, Binary comonotonic act independence, means that the preference relation restricted to acts that are measurable with respect to two-element partitions, conforms to the theory of CEU. With these four axioms, Ghirardato and Marinacci [12] were able to prove that the preference relation admits what they dubbed a (canonical) biseparable representation, namely, a CEU representation defined on this restricted set of acts.

Proposition 5.1 (*Ghirardato and Marinacci [12, Theorem 11]*). *Let X be a connected and separable topological space and let $\succsim$ be a non-trivial binary relation on $\mathcal{F}$. Then the following statements are equivalent*:

(i) $\succsim$ *satisfies Axioms* 1–4.
(ii) *There exists a unique subcontinuous non-trivial monotonic representation of* $\succsim$, $V : \mathcal{F} \to \mathbb{R}$, *and a unique capacity* $v : \mathcal{E} \to [0, 1]$ *such that for all* $x \succsim y$ *and for all* $E \in \mathcal{E}$

$$V(x_E y) = v(E)\, u(x) + (1 - v(E))\, u(y), \tag{3}$$

where $u(x) := V(x)$ *for all* $x \in X$. *Moreover*, V *is unique up to positive affine transformations.*

Statement (ii) of Proposition 5.1 says that the preference relation $\succsim$ on $\mathcal{F}$ is biseparable in the sense of Definition 1 in GMMS [11].

It remains to impose an appropriate version of an independence-type axiom that extends the biseparable CEU representation obtained in Proposition 5.1 to the whole domain $\mathcal{F}$ and, moreover, entails that the capacity in that representation is neo-additive. To do this, we first need to define GMMS's [11] notion of a 'subjective mixture' of two acts. We begin with their definition of a 'preference average' of two consequences.

Definition 5.1. Fix $x, y \in X$, such that $x \succ y$. We say that a consequence $z \in X$ is a *preference average* of x and y (given E) if $x \succsim z \succsim y$ and

$$x_E y \sim m(x_E z)_E\, m(z_E y).$$

The reason for their nomenclature becomes apparent if we consider for a preference relation that satisfies Axioms 1–4, the preference average of x and y given an essential event E. From Proposition 5.1 we obtain the equality

$$\begin{aligned} & v(E)\, u(x) + (1 - v(E))\, u(y) \\ & \quad = v(E)\, u(m(x_E z)) + (1 - v(E))\, u(m(z_E y)) \\ & \quad = v(E)^2\, u(x) + 2v(E)(1 - v(E))\, u(z) + (1 - v(E))^2\, u(y). \end{aligned}$$

Notice that if an event E is essential then $0 < v(E) < 1$, and so solving for $u(z)$ yields

$$u(z) = \tfrac{1}{2} u(x) + \tfrac{1}{2} u(y), \tag{4}$$

which is independent of E. We shall therefore denote by $(\frac{1}{2})x \oplus (\frac{1}{2})y$ the preference average of the outcomes x and y.

Our next axiom is key to characterizing the decision maker's attitudes towards events that yield extreme outcomes. We first need, however, to define for each act which events the decision maker views as yielding the extreme outcomes. We begin with preference-based definitions for the infimum and the supremum of an act.

Definition 5.2. Fix $f \in \mathcal{F}$. An outcome $z \in X$ is said to be in the indifference set of the infimum of f, $z \in \inf_{\succcurlyeq}(f)$, if for $A := f^{-1}(x : z \succ x)$, $z_A f \sim f$ and if for every $y \succ z$ and $B := f^{-1}(x : y \succcurlyeq x)$, $y_B f \succ f$. Similarly, an outcome $z \in X$ is said to be in the indifference set of the supremum of f, $z \in \sup_{\succcurlyeq}(f)$ if for $A := f^{-1}(x : x \succ z)$, $z_A f \sim f$ and if for every y such that $z \succ y$ and $B := f^{-1}(x : x \succcurlyeq y)$, $f \succ y_B f$.

Although $\inf_{\succcurlyeq}(f)$ and $\sup_{\succcurlyeq}(f)$ are defined to be indifference sets of outcomes, when there is no risk of confusion, we shall write $\inf_{\succcurlyeq}(f)$ and $\sup_{\succcurlyeq}(f)$ to indicate arbitrary members of these respective sets.

From the definition of a subjective mixture and Eq. (4) it follows that for every $f, g \in \mathcal{F}$ and $s \in S$

$$u\left(\tfrac{1}{2}f(s) \oplus \tfrac{1}{2}g(s)\right) = \tfrac{1}{2}u(f(s)) + \tfrac{1}{2}u(g(s)) \tag{5}$$

is well-defined. We call the act $\frac{1}{2}f \oplus \frac{1}{2}g$, obtained by the statewise mixture of Eq. (5), a *subjective mixture* of the two acts f and g.

If there is a non-null event E on which both the acts f and g attain their supremum,

$$E = \left\{s \in S : f(s) \in \sup_{\succcurlyeq}(f)\right\} \cap \left\{s \in S : g(s) \in \sup_{\succcurlyeq}(g)\right\} \notin \mathcal{N},$$

then a subjective mixture of these two acts will also attain its supremum on this set, i.e., for all $s \in E$, $\frac{1}{2}f \oplus \frac{1}{2}g(s) \in \sup_{\succcurlyeq}(\frac{1}{2}f \oplus \frac{1}{2}g)$. If the set E is null, then a subjective mixture of these acts need not attain its supremum on any state in $\left\{s \in S : f(s) \succsim \sup_{\succcurlyeq}(f)\right\}$ nor on any state in $\left\{s \in S : g(s) \succsim \sup_{\succcurlyeq}(g)\right\}$. Similarly, if the set of states F, on which either f or g yield outcomes which are strictly preferred to the $\inf_{\succcurlyeq}(f)$ and $\inf_{\succcurlyeq}(g)$, respectively,

$$F = \left\{s \in S : f(s) \succ \inf_{\succcurlyeq}(f)\right\} \cup \left\{s \in S : g(s) \succ \inf_{\succcurlyeq}(g)\right\} \notin \mathcal{U}$$

is not universal, then $\frac{1}{2}f \oplus \frac{1}{2}g(s) \in \inf_{\succcurlyeq}(\frac{1}{2}f \oplus \frac{1}{2}g)$ for all $s \in S \backslash F$.

The idea behind the next axiom is that if the decision maker is sensitive about extreme events, both for good and bad outcomes, then subjective mixtures of acts for which supremum outcomes are obtained on common non-null events will be viewed relatively favorably while subjective mixtures of acts for which infimum outcomes are obtained on a common event whose complement is not universal will be viewed relatively unfavorably.

To facilitate the formulation of this axiom, define an act h to be a member of $\underline{\mathcal{F}}(f)$, if the set of states on which a non-infimum outcome obtains either for h or for f is not universal. That is,

$$\underline{\mathcal{F}}(f) := \left\{h \in \mathcal{F} \,\middle|\, \left\{s \in S : f(s) \succ \inf_{\succcurlyeq}(f)\right\} \cup \left\{s \in S : h(s) \succ \inf_{\succcurlyeq}(h)\right\} \notin \mathcal{U}\right\}.$$

Similarly, denote by $\overline{\mathcal{F}}(f)$ the set of acts with some supremum outcomes on a common non-null event as the act f. That is,

$$\overline{\mathcal{F}}(f) := \left\{ h \in \mathcal{F} \,\middle|\, \left\{ s \in S : f(s) \underset{\succsim}{\succsim} \sup(f) \right\} \cap \left\{ s \in S : h(s) \underset{\succsim}{\succsim} \sup(h) \right\} \notin \mathcal{N} \right\}.$$

In GMMS [11]'s axiomatization of CEU, their key axiom is the restriction of an independence type axiom to subjective mixtures of comonotonic acts. The Choquet integral of a neo-additive capacity satisfies the independence axiom for all acts with the best and worst outcomes on the same events, respectively. Hence, we effectively strengthen their comonotonic independence axiom by requiring it to hold for all acts which obtain their infimum (and, respectively, their supremum) on some common non-null event.

Axiom 5 (*Extreme events sensitivity*). For any $f, g, h \in \mathcal{F}$ such that $f \sim g$ and $h \in \underline{\mathcal{F}}(g) \cap \overline{\mathcal{F}}(g)$,

1. if $h \in \underline{\mathcal{F}}(f)$ then $\frac{1}{2}g \oplus \frac{1}{2}h \succsim \frac{1}{2}f \oplus \frac{1}{2}h$,
2. if $h \in \overline{\mathcal{F}}(f)$ then $\frac{1}{2}f \oplus \frac{1}{2}h \succsim \frac{1}{2}g \oplus \frac{1}{2}h$.

Figs. 1 and 2 illustrate the behavioral assumptions embodied in Axiom 5. Consider the special case where $h = g$. Hence, $h \in \underline{\mathcal{F}}(g) \cap \overline{\mathcal{F}}(g)$ is trivially satisfied. Fig. 1 shows two acts f and h with $h \in \underline{\mathcal{F}}(f)$.

Axiom 5 requires that a decision maker who is indifferent between f and g weakly prefers a subjective half–half mixture of g and h over the subjective half–half mixture of f and h. We refer to this case as 'optimism' because the subjective mixture of f and h will in general 'hedge' the 'utility' on the 'good' events without changing the utility on the worst event.

Analogously, when $h \in \overline{\mathcal{F}}(f)$ the decision maker should weakly prefer the act subjective half–half mixture of f and h to the half–half subjective mixture of g and h, since the subjective mixture of f and h now leaves the utility on the best event unchanged but 'hedges' the 'utility' on the 'bad' events. This case is illustrated in Fig. 2. Here the mixture leaves the event with the worst outcome unchanged.

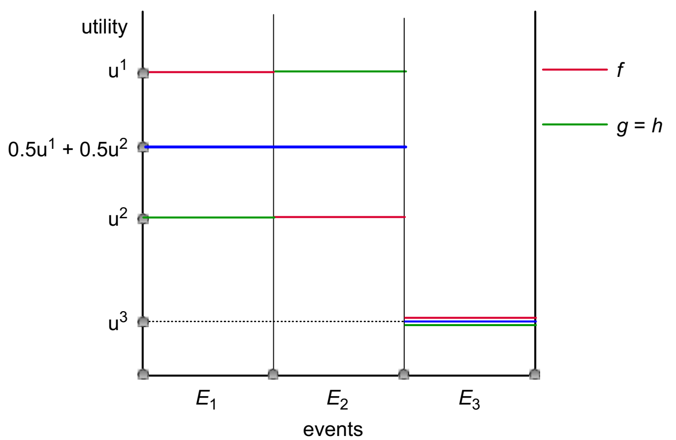

Fig. 1. Illustration of Axiom 5—optimism: $h \in \underline{\mathcal{F}}(f)$ and so $(h =)\frac{1}{2}g \oplus \frac{1}{2}h \succsim \frac{1}{2}f \oplus \frac{1}{2}h$.

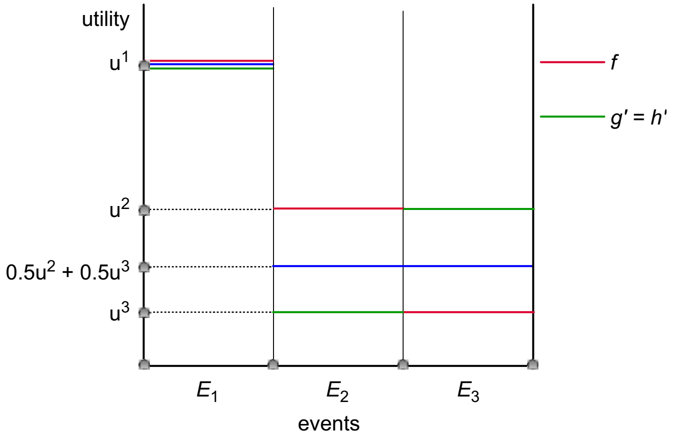

Fig. 2. Illustration of Axiom 5—Pessimism: $h' \in \overline{\mathcal{F}}(f)$ and so $\frac{1}{2}f \oplus \frac{1}{2}h' \succsim \frac{1}{2}g' \oplus \frac{1}{2}h'(= h')$.

Acts that are comonotonic with f are elements of $\underline{\mathcal{F}}(f) \cap \overline{\mathcal{F}}(f)$. Hence Axiom 5 implies comonotonic independence for the indifference relation. In the proof of our representation result, we show that in conjunction with the other axioms it characterizes a subclass of the family of CEU functionals.

Axiom 5, the key axiom for the neo-additive capacity representation, is testable in experiments. One needs only to check the choice of a decision maker over acts which represent half–half subjective mixtures. This is substantially less demanding than testing independence for arbitrary mixtures of acts.

The next axiom assumes the individual envisions the set $\mathcal{N}$ of null events and the set $\mathcal{U}$ of universal events in a consistent way. In particular, it states that any event E on which the individual is unwilling to bet, he is willing to bet against, or equivalently, if an event E is null then its complement $S \backslash E$ is universal. [7]

Axiom 6 (*Null event consistency*). For all pairs of outcomes x and y such that $x \succ y$, $x_E y \sim y$ implies $y_E x \sim x$.

The following theorem contains our main result. Recall that we call a capacity v *non-trivial* if there exists an event E such that $v(E) \in (0, 1)$.

Theorem 5.1. *Let X be a connected and separable topological space, and let $\succsim$ be a non-trivial binary preference relation on $\mathcal{F}$. If $\mathcal{E}^*$ contains at least four disjoint elements, then the following two statements are equivalent:*

1. *The preference relation $\succsim$ on $\mathcal{F}$ satisfies* Ordering, Continuity, Eventwise monotonicity, Binary comonotonic act independence, Extreme events sensitivity *and* Null event consistency.

[7] Notice that Axiom 6 implies that null and universal sets in the sense of Ghirardato and Marinacci [12] are also null and universal in the sense of Savage [26].

2. *There exists a unique non-trivial* neo-additive capacity v *on* $\mathcal{E}$ *and a unique continuous real-valued function* u *on* X, *with* $u(0) = 0$ *and* $u(M) = 1$, *such that for all* $f, g \in \mathcal{F}$

$$f \succsim g \quad \Leftrightarrow \int u \circ f \, dv \geqslant \int u \circ g \, dv.$$

Remark. Comparing the conditions of Proposition 3.1 and Theorem 5.1, one may wonder why the event space $\mathcal{E}$ was required to have *at least three non-intersecting events* that are essential in the former result but had to have *at least four such events* in the latter. In Theorem 5.1 four non-intersecting essential events are necessary in order to prove that the capacity v which we deduce satisfies statement (ii a) of Proposition 3.1. Appendix A.6 contains a formal argument.

The proof of Theorem 5.1 proceeds in two steps. In a first step we show that our axioms imply the existence of a Choquet representation. The second step verifies that Axiom 5 yields the properties listed in Proposition 3.1 (ii). Hence, our capacity is neo-additive. The first part of the proof requires, however, only comonotonic independence for the indifference relation. In so far, the Choquet representation result holds under weaker conditions than Proposition 2 in GMMS [11].

6. Concluding comments

In this paper we have introduced a special case of capacity and its Choquet integral which captures aspects of optimism and pessimism without abandoning the subjectively probabilistic approach all together. In particular, subjective expected utility is always contained as a special parametric case in this approach. Moreover, as in Eichberger and Kelsey [7], the additive part of a neo-additive capacity can be determined endogenously in equilibrium.

Most importantly neo-additive capacities open new avenues of research. It appears natural to view the degree of confidence which a decision maker holds in a probabilistic assessment of an uncertain situation as dependent on past experience and subject to influence from other people's beliefs. Optimism and pessimism may spread in a population. Attitudes towards uncertain outcomes may be contagious leading to general swings in optimism and pessimism. So 'irrational exuberance' as observed by Shiller [27] may become amenable to formal economic analysis after all.

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Appendix A. Proofs

Throughout this appendix we will denote by $\mathcal{E}^*$ the set of essential events and assume the existence of at least three pairwise disjoint essential events denoted by A_1, A_2, A_3.

A.1. Properties of essential events

Property 1. $E \in \mathcal{E}^*$ implies $S \backslash E \in \mathcal{E}^*$.

Proof. The sets $\mathcal{N}$, $\mathcal{E}^*$, and $\mathcal{U}$ partition the set of events $\mathcal{E}$. Let $E \in \mathcal{E}^*$, then $S \backslash E \in \mathcal{U}$ is impossible, since otherwise $E \in \mathcal{N}$. $S \backslash E \in \mathcal{N}$ is impossible, because otherwise $E \in \mathcal{U}$. Hence, $S \backslash E \in \mathcal{E}^*$ must be true. $\square$

Property 2. There exists a partition F_1, F_2, F_3 of S such that $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$.

Proof. Let $F_1 = A_1$, $F_2 = A_2$. It remains to show that $F_3 = S \backslash (A_1 \cup A_2) \in \mathcal{E}^*$. Suppose $F_3 \in \mathcal{U}$. This implies $A_1 \cup A_2 \in \mathcal{N}$, and, hence, $A_1, A_2 \in \mathcal{N}$. Similarly, assume $F_3 \in \mathcal{N}$. This implies $A_3 \subset F_3$ and, hence, $A_3 \in \mathcal{N}$. Since $A_1, A_2, A_3 \in \mathcal{E}^*$, $F_3 = S \backslash (A_1 \cup A_2) \in \mathcal{E}^*$ must be true. $\square$

Property 3. Let $E_1, E_2, E_3 \in \mathcal{E}^*$ be pairwise disjoint sets, then $E_i \cup E_j \in \mathcal{E}^*$ for $i, j \in 1, 2, 3$.

Proof. The claim follows immediately from Property 1. $\square$

Property 4. Let $E \in \mathcal{E}^*$, then there exists a partition E_1, E_2, E_3 of S with $E_i \in \mathcal{E}^*$, for $i = 1, 2, 3$, such that either $E = E_1 \cup E_2$ or $S \backslash E = E_2 \cup E_3$.

Proof. Let $E \in \mathcal{E}^*$. From Property 1, $S \backslash E \in \mathcal{E}^*$. Property 2 guarantees that there is a partition F_1, F_2, F_3 of S such that $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$. Let $E_i' = E \cap F_i$ and $E_i'' = (S \backslash E) \cap F_i$, then $E_i' \cup E_i'' = F_i$. Hence, neither $E_i' \in \mathcal{U}$ nor $E_i'' \in \mathcal{U}$, and either $E_i' \notin \mathcal{N}$ or $E_i'' \notin \mathcal{N}$. Therefore, either $E_i' \in \mathcal{E}^*$ or $E_i'' \in \mathcal{E}^*$. Moreover, there is $i, j \in \{1, 2, 3\}$ such that $E_i', E_j'' \in \mathcal{E}^*$, for otherwise either $E \in \mathcal{U}$ or $S \backslash E \in \mathcal{U}$ contradicting Property 1. $\square$

A.2. *Uniqueness of the parameters* (π, α, δ)

Lemma A.1. *Let $\mathcal{N}$ be a given set of null events and let $A_1, A_2, A_3 \in \mathcal{E}^*$ be pairwise disjoint events. If $v(E|\mathcal{N}, \pi, \delta, \alpha)$ is a neo-additive capacity on $(S, \mathcal{E})$ which is congruent with $\mathcal{N}$, then (π, α, δ) are unique.*

Proof. By Property 2, there is a partition F_1, F_2, F_3 of S such that $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$. Let (π, α, δ) be a given vector of parameters. We will prove that it is unique. Denote by $\widehat{\pi}(E) = (1 - \delta)\pi(E)$.

(i) The product $\alpha\delta$ is uniquely defined: Property 3 and Definition 3.3 imply $v(F_1 \cup F_2) = \widehat{\pi}(F_1 \cup F_2) + \alpha\delta$, $v(F_1) = \widehat{\pi}(F_1) + \alpha\delta$, $v(F_2) = \widehat{\pi}(F_2) + \alpha\delta$. Additivity of $\widehat{\pi}$ implies

$$\alpha\delta = v(F_1) + v(F_2) - v(F_1 \cup F_2).$$

By symmetry, we have also $\alpha\delta = v(F_1) + v(F_3) - v(F_1 \cup F_3)$ and $\alpha\delta = v(F_2) + v(F_3) - v(F_2 \cup F_3)$.

(ii) $(1 - \delta)$ is uniquely defined: Since π is a probability distribution, $\pi(F_1) + \pi(F_2) + \pi(F_3) = 1$. Hence, $\widehat{\pi}(F_1) + \widehat{\pi}(F_2) + \widehat{\pi}(F_3) = 1 - \delta$.

Moreover, from step (i), $\widehat{\pi}(F_i) = v(F_i) - \alpha\delta$ and $\widehat{\pi}(F_i \cup F_j) = v(F_i \cup F_j) - \alpha\delta$ for all $i, j \in \{1, 2, 3\}$. This implies $\widehat{\pi}(F_i) = v(F_i \cup F_j) - v(F_j)$. Summing, we obtain

$$1 - \delta = v(F_1 \cup F_2) + v(F_2 \cup F_3) + v(F_1 \cup F_3) - v(F_1) - v(F_2) - v(F_3).$$

(iii) π is uniquely defined: If $\delta = 1$, then $v(E|\mathcal{N}, \pi, \delta, \alpha) = \mu_\alpha^{\mathcal{N}}(E)$ and π need not be defined. If $\delta < 1$, then, for $E \in \mathcal{E}^*$, $v(E) = \widehat{\pi}(E) + \alpha\delta$ defines $\widehat{\pi}(E) = v(E) - \alpha\delta$ uniquely. Since π

is congruent with $\mathcal{N}$, one has $\widehat{\pi}(E) = 0$ for $E \in \mathcal{N}$ and $\widehat{\pi}(E) = 1 - \delta$ for $E \in \mathcal{U}$. Hence, π is uniquely defined for all $E \in \mathcal{E}$ by

$$\pi(E) = \frac{\widehat{\pi}(E)}{1-\delta}.$$

(iv) α is uniquely defined: If $\delta = 1$, then $v(E|\mathcal{N}, \pi, \delta, \alpha) = \mu_\alpha^{\mathcal{N}}(E)$ and α is uniquely defined as

$$\alpha = v(F_1) + v(F_2) - v(F_1 \cup F_2).$$

If $0 < \delta < 1$, then, from step (i),

$$\alpha = \frac{v(F_1) + v(F_2) - v(F_1 \cup F_2)}{\delta}$$

defines α uniquely. For $\delta = 0$, $v(E|\mathcal{N}, \pi, \delta, \alpha) = \pi(E)$ and α needs not be defined. $\square$

A.3. Proof of Proposition 3.1

(i) $\Longrightarrow$ (ii). This follows from the definition of a neo-additive capacity.

(ii) $\Longrightarrow$ (i).

(a) We define a non-negative measure $\widehat{\pi}$ on $(S, \mathcal{E})$.

Case I: $E \in \widehat{\mathcal{E}} = \{E \in \mathcal{E}^* | \exists F \in \mathcal{E}^*, E \cap F = \emptyset E \cup F \in \mathcal{E}^*\}$.

Define $\widehat{\pi}(E) = v(E \cup F) - v(F)$ for all $F \in \mathcal{E}^*$, $E \cap F = \emptyset$ $E \cup F \in \mathcal{E}^*$. By Property 1, is well-defined and, by monotonicity of v, $\widehat{\pi}(E) \geqslant 0$ for all $E \in \widehat{\mathcal{E}}$.

Case II: $E \in \mathcal{E}^* \backslash \widehat{\mathcal{E}}$.

From Properties 3 and 4, $E \in \mathcal{E}^* \backslash \widehat{\mathcal{E}}$ implies $S \backslash E \in \widehat{\mathcal{E}}$. Define $\widehat{\pi}(E) = v(E) + \widehat{\pi}(S \backslash E) - v(S \backslash E)$. By Property 4 and Case I, we have $\widehat{\pi}(E) = v(E_1 \cup E_2) + \widehat{\pi}(E_3) - v(E_3) = v(E_1 \cup E_2) + [v(E_2 \cup E_3) - v(E_2)] - v(E_3) = [v(E_1 \cup E_2) - v(E_2)] + [v(E_2 \cup E_3) - v(E_3)] \geqslant 0$ by monotonicity.

Case III: $E \in \mathcal{N}$. Let $\widehat{\pi}(E) = 0$.

Case IV: $E \in \mathcal{U}$. Define $\widehat{\pi}(E) = \widehat{\pi}(F_1) + \widehat{\pi}(F_2) + \widehat{\pi}(F_3)$ for a fixed partition F_1, F_2, F_3 of S with $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$. Such a partition exists by Property 2.

(b) We show that $\widehat{\pi}$ is additive on $(S, \mathcal{E})$, $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$ for all disjoint E_1, $E_2 \in \mathcal{E}$.

Case 1: $E_1, E_2 \in \mathcal{E}^*$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \in \mathcal{E}^*$.

Case 1.1: $E = E_1 \cup E_2 \in \widehat{\mathcal{E}}$. Hence, there exists $F \in \mathcal{E}^*$, $E \cap F = \emptyset$ $E \cup F \in \mathcal{E}^*$. By Case I, $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_2 \cup F) - v(F) = [v(E_1 \cup E_2 \cup F) - v(E_2 \cup F)] + [v(E_2 \cup F) - v(F)]$. From Property 3, $E_2 \cup F \in \mathcal{E}^*$ and, hence, $E_1, E_2 \in \widehat{\mathcal{E}}$. Thus, we can apply Case I, $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$.

Case 1.2: $E = E_1 \cup E_2 \in \mathcal{E}^* \backslash \widehat{\mathcal{E}}$.

Let $E_3 = S \backslash E$. From Case II, $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_2) + \widehat{\pi}(E_3) - v(E_3)$. Since $E_3 \in \widehat{\mathcal{E}}$ and $E_3 \cup E_2 \in \mathcal{E}^*$, we have $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_2) + [v(E_2 \cup E_3) - v(E_2)] - v(E_3) = [v(E_1 \cup E_2) - v(E_2)] + [v(E_2 \cup E_3) - v(E_3)]$. Thus, $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$ because $E_1, E_2 \in \widehat{\mathcal{E}}$.

Case 2: $E_1 \in \mathcal{E}^*$, $E_2 \in \mathcal{N}$ and $E_1 \cup E_2 \in \mathcal{E}^*$.

Case 2.1: $E_1 \cup E_2 \in \widehat{\mathcal{E}}$.

There exists $F \in \mathcal{E}^*$ such that $F \cap (E_1 \cup E_2) = \emptyset$ and $F \cup (E_1 \cup E_2) \in \mathcal{E}^*$. From Case I, $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_2 \cup F) - v(F)$. From $E_2 \in \mathcal{N}$ and $F \cup (E_1 \cup E_2) \in \mathcal{E}^*$, we conclude $E_1 \cup F \in \mathcal{E}^*$ and, hence, $E_1 \in \widehat{\mathcal{E}}$. Thus, applying Case I, $\widehat{\pi}(E_1) = v(E_1 \cup F) - v(F)$. From property (d) of Proposition 3.1, we have $v(E_1 \cup E_2 \cup F) = v(E_1 \cup F)$ and, hence, $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_2 \cup F) - v(F) = v(E_1 \cup F) - v(F)$. Hence, because $\widehat{\pi}(E_2) = 0$ by Case III, we have $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$.

Case 2.2: $E_1 \cup E_2 \in \mathcal{E}^* \backslash \widehat{\mathcal{E}}$.

From $E_2 \in \mathcal{N}$, we have $E_1 \cup E_3 \in \mathcal{E}^*$ where $E_3 = S \backslash (E_1 \cup E_2)$. Hence, as in Case 1.2 $\widehat{\pi}(E_1 \cup E_2) = [v(E_1 \cup E_2) - v(E_2)] + [v(E_1 \cup E_3) - v(E_3)]$. From property (d) of Proposition 3.1, we have $v(E_1 \cup E_2) = v(E_1)$ and, hence, using Case III, $\widehat{\pi}(E_1 \cup E_2) = v(E_1 \cup E_3) - v(E_3) = \widehat{\pi}(E_1) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$.

Case 3: $E_1, E_2 \in \mathcal{N}$.

Since $E_1 \cup E_2 \in \mathcal{N}$, $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$ follows immediately from Case III.

Case 4: $E_1, E_2 \in \mathcal{E}^*$ and $E_1 \cup E_2 \in \mathcal{U}$.

From Case IV, we know that $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(F_1) + \widehat{\pi}(F_2) + \widehat{\pi}(F_3)$ for a fixed partition F_1, F_2, F_3 of S, with $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$. Since $E_i \in \mathcal{E}^*$, either $E_i \cap F_j \in \mathcal{E}^*$ or $E_i \cap F_j \in \mathcal{N}$ must be true for $i = 1, 2$ and $j = 1, 2, 3$. Hence, from Cases 1 to 3, we conclude $\widehat{\pi}(E_i) = \widehat{\pi}(E_i \cap F_1) + \widehat{\pi}(E_i \cap F_2) + \widehat{\pi}(E_i \cap F_3)$ for $i = 1, 2$. Since $E_1 \cup E_2 \in \mathcal{U}$ holds, $E_3 = S \backslash (E_1 \cup E_2) \in \mathcal{N}$ follows. Hence, for $j = 1, 2, 3$, $E_3 \cap F_j \in \mathcal{N}$ and, by Case III, $\widehat{\pi}(E_3 \cap F_j) = 0$. Thus, $\widehat{\pi}(E_1 \cup E_2) = \sum_{j=1}^{3} \widehat{\pi}(F_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} \widehat{\pi}(E_i \cap F_j) = \sum_{j=1}^{3} \widehat{\pi}(E_1 \cap F_j) + \sum_{j=1}^{3} \widehat{\pi}(E_2 \cap F_j) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$.

Case 5: $E_1 \in \mathcal{E}^*$, $E_2 \in \mathcal{N}$ and $E_1 \cup E_2 \in \mathcal{U}$.

Additivity follows from the same arguments as in Case 4.

Case 6: $E_1 \in \mathcal{U}$ and $E_2 \in \mathcal{N}$.

Since $E_1 \cup E_2 \in \mathcal{U}$, $\widehat{\pi}(E_1 \cup E_2) = \widehat{\pi}(E_1) + \widehat{\pi}(E_2)$ follows immediately from Cases III and IV.

All other cases are impossible.

(b) We prove that there exists $\lambda \geqslant 0$ such that $v(E) = \lambda + \widehat{\pi}(E)$ for all $E \in \mathcal{E}^*$.

(b1) We prove that $\lambda = v(E) - \widehat{\pi}(E)$ for all $E \in \mathcal{E}^*$.

Consider the fixed partition of S from Case IV, F_1, F_2, F_3 of S with $F_i \in \mathcal{E}^*$ for $i = 1, 2, 3$. Let $\lambda = v(F_1) - \widehat{\pi}(F_1)$. By Property 3 and (a), we have $\lambda = v(F_1) - \widehat{\pi}(F_1) = v(F_1) - [v(F_1 \cup F_j) - v(F_j)] = v(F_j) - [v(F_1 \cup F_j) - v(F_1)] = v(F_j) - \widehat{\pi}(F_j)$ for $j = 2, 3$.

We will show now that $\lambda = v(F_1) - \widehat{\pi}(F_1) = v(E) - \widehat{\pi}(E)$.

Case b1.1: Suppose $E \in \widehat{\mathcal{E}}$. Since $E = \bigcup_{j=1}^{3} (E \cap F_j) \in \mathcal{E}^*$, at least one of the events $E \cap F_j$ must be essential. W.l.o.g., assume $E \cap F_1 \in \mathcal{E}^*$.

We show first that $\lambda = v(F_1) - \widehat{\pi}(F_1) = v(E \cap F_1) - \widehat{\pi}(E \cap F_1)$. By Property 4 there exists a partition of S into three essential events $E, F, G \in \mathcal{E}^*$. If $F \cap F_1 \in \mathcal{N}$, then

$$\begin{aligned} v(F_1) - \widehat{\pi}(F_1) &= v((E \cap F_1) \cup (F \cap F_1) \cup (G \cap F_1)) - \widehat{\pi}((E \cap F_1) \\ &\quad \cup (F \cap F_1) \cup (G \cap F_1)) \\ &= v((E \cap F_1) \cup (G \cap F_1)) - \widehat{\pi}((E \cap F_1) \cup (G \cap F_1)) \\ &= [v((E \cap F_1) \cup (G \cap F_1)) - v(E \cap F_1)] + v(E \cap F_1) - \widehat{\pi}((E \cap F_1)) \\ &\quad - \widehat{\pi}((G \cap F_1)) \\ &= v(E \cap F_1) - \widehat{\pi}((E \cap F_1)) \end{aligned}$$

by Property 4, since $(E \cap F_1) \cup (G \cap F_1) \in \mathcal{E}^*$, and Case IV.

If $F \cap F_1 \in \mathcal{E}^*$, then Case I implies $\widehat{\pi}((E \cap F_1)) = v(F_1) - v((E \cap F_1) \cup (G \cap F_1))$ and, hence,

$$\begin{aligned} v(F_1) - \widehat{\pi}(F_1) &= v((E \cap F_1) \cup (G \cap F_1)) + \widehat{\pi}(F \cap F_1) - \widehat{\pi}(F_1) \\ &= v((E \cap F_1) \cup (G \cap F_1)) - \widehat{\pi}((E \cap F_1) \cup (G \cap F_1)) \\ &= v(E \cap F_1) - \widehat{\pi}(E \cap F_1). \end{aligned}$$

Next we show that $v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E)$. Consider four cases.

Case b1.1a: $(E \cap F_2) \cup (E \cap F_3) \in \mathcal{N}$. By Property (ii d) of Proposition 3.1 and additivity of $\widehat{\pi}$, we have $v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E)$ and, hence,

$$\lambda = v(F_1) - \widehat{\pi}(F_1) = v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E).$$

Case b1.1b: $(E \cap F_2) \in \mathcal{N}$ and $(E \cap F_3) \in \mathcal{E}^*$.

$$v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v((E \cap F_1) \cup (E \cap F_2)) - \widehat{\pi}((E \cap F_1) \cup (E \cap F_2)).$$

From Case I, we have $\widehat{\pi}(E \cap F_3) = v(E) - v((E \cap F_1) \cup (E \cap F_2))$. Hence, by additivity of $\widehat{\pi}$,

$$v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E).$$

Case b1.1c: For the symmetric case $(E \cap F_2) \in \mathcal{E}^*$ and $(E \cap F_3) \in \mathcal{N}$, the same argument applies.

Case b1.1d: $(E \cap F_2) \in \mathcal{E}^*$ and $(E \cap F_3) \in \mathcal{E}^*$.

Applying Case I twice, yields

$$v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v((E \cap F_1) \cup (E \cap F_2)) - \widehat{\pi}((E \cap F_1) \cup (E \cap F_2))$$

and

$$v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E).$$

Thus, we have $\lambda = v(F_1) - \widehat{\pi}(F_1) = v(E \cap F_1) - \widehat{\pi}(E \cap F_1) = v(E) - \widehat{\pi}(E)$.

Case b1.2: Suppose $E \in \mathcal{E}^* \backslash \widehat{\mathcal{E}}$. From Case II, we have $v(E) - \widehat{\pi}(E) = v(S \backslash E) - \widehat{\pi}(S \backslash E)$. Since $S \backslash E \in \widehat{\mathcal{E}}$, Case b1.1 implies $\lambda = v(E) - \widehat{\pi}(E)$.

(b2) We prove that $\lambda \geqslant 0$.

By property (b) of Proposition 3.1, there exist $E, F \in \mathcal{E}^*$ with $E \cap F = \emptyset$ and $E \cup F \in \mathcal{E}^*$ such that $v(E) + v(F) - v(E \cup F) \geqslant 0$. Since $E \in \widehat{\mathcal{E}}$ holds, we have $\lambda = v(E) - \widehat{\pi}(E) = v(E) - [v(E \cup F) - v(F)] = v(E) + v(F) - v(E \cup F) \geqslant 0$.

(c) Define $\delta = 1 - \widehat{\pi}(S)$. Since $\widehat{\pi}(S) \geqslant 0$, we know $\delta \leqslant 1$. By property (c) of Proposition 3.1, there exist $E, F \in \mathcal{E}^*$ with $E \cap F = \emptyset$ and $E \cup F \in \mathcal{E}^*$ such that $\overline{v}(E) + \overline{v}(F) - \overline{v}(E \cup F) \geqslant 0$. Recalling from part (b) of this proof that $v(E) = \lambda + \widehat{\pi}(E)$, one obtains by straightforward computations:

$$\begin{aligned} 0 &\leqslant \overline{v}(E) + \overline{v}(F) - \overline{v}(E \cup F) = 1 - v(S \backslash E) - v(S \backslash F) + v(S \backslash E \cap S \backslash F) \\ &= 1 - \left[\lambda + \widehat{\pi}(S \backslash E)\right] - \left[\lambda + \widehat{\pi}(S \backslash F)\right] + \left[\lambda + \widehat{\pi}(S \backslash E \cap S \backslash F)\right] \\ &= 1 - \lambda - \widehat{\pi}(S \backslash E \cup S \backslash F) = 1 - \lambda - (1 - \delta) = \delta - \lambda. \end{aligned}$$

Note that $\lambda \geqslant 0$ and, hence, $\delta \in [0, 1]$.

(d) For $\delta \in (0,1)$, define $\alpha = \frac{\lambda}{\delta}$ and $\pi(E) := \frac{\widehat{\pi}(E)}{1-\delta}$ for all $E \in \mathcal{E}$. With these definitions, we have

$$v(E) = \begin{cases} 0 & \text{if } E \in \mathcal{N}, \\ \delta\alpha + (1-\delta)\pi(E) & \text{if } E \in \mathcal{E}^*, \\ 1 & \text{if } E \in \mathcal{U}. \end{cases}$$

Hence, $v(E) := (1-\delta)\,\pi(E) + \delta\mu_\alpha^{\mathcal{N}}(E)$.

For $\delta = 0$, we have $\widehat{\pi}(S) = 1$ and, by part (c) of this proof, $\lambda = \delta\alpha = 0$.

Hence, $v(E) := \pi(E) = \widehat{\pi}(E)$ for all $E \in \mathcal{E}$ defines an additive probability distribution.

Finally, for $\delta = 1$, one has $\alpha = \lambda$ and $v(E) = \mu_\alpha^{\mathcal{N}}(E)$.

A.4. *Proof of Lemma 4.1*

From Eq. (2), we get

$$\begin{aligned} q^* &= \frac{\delta \cdot u'(\underline{r} \cdot A) \cdot \underline{r} + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \left[\delta \cdot u'(\underline{r} \cdot A) + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} \\ &= \frac{\delta \cdot u'(\underline{r} \cdot A) \cdot \underline{r} + (1-\delta) \cdot \left[r \cdot q_0^* \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}{r \cdot \left[\delta \cdot u'(\underline{r} \cdot A) + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} \\ &< \frac{\delta \cdot u'(\underline{r} \cdot A) \cdot r \cdot q_0^* + (1-\delta) \cdot \left[r \cdot q_0^* \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}{r \cdot \left[\delta \cdot u'(\underline{r} \cdot A) + (1-\delta) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} = q_0^*. \end{aligned}$$

The inequality follows because

$$r \cdot q_0^* = \frac{\sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{\sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)} > \underline{r}.$$

A.5. *Proof of Theorem 5.1*

We begin with an observation and a couple of preliminary results. The observation is that any act $f \in \mathcal{F}$ may be expressed as $[x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$, where $\{E_1, \ldots, E_n\}$ is the coarsest finite $\succsim$-ordered partition of S with respect to which f is measurable. By that we mean for any pair of states $s, t \in S$, if both s and t are in some $E \in \{E_1, \ldots, E_n\}$ then $f(s) = f(t)$, otherwise $f(s) \neq f(t)$. Furthermore, for any $s \in E_i$ and $t \in E_j$, $i < j$ implies $f(s) \succsim f(t)$. Throughout this proof, if an act is expressed in the form $[x_1 \text{ on } E_1; \ldots;\ x_n \text{ on } E_n]$ then it should be taken as given that $x_i \succsim x_{i+1}$, for $i = 1, \ldots, n-1$.

From Eq. (4) we know that a preference average $\frac{1}{2}x \oplus \frac{1}{2}y$ of the outcomes x and y is well-defined by $x_E y \sim m(x_E z)_E\, m(z_E y)$ for an arbitrary essential event E. To derive weighted averages $\alpha x \oplus (1-\alpha) y$ of x and y for arbitrary $\alpha \in [0,1]$, we follow the line of argument detailed in GMMS [11]. By using iterated averages and appealing to standard continuity arguments, it is possible to identify, for any $\alpha \in [0,1]$ and every $x, y \in X$, preference averages characterized by

$$u(z) = \alpha u(x) + (1-\alpha)\, u(y) \tag{A.1}$$

With slight abuse of notation, we shall let $\alpha x \oplus (1-\alpha)\, y$ (or, equivalently, $(1-\alpha)\, y \oplus \alpha x$) denote an arbitrary element of the indifferent set of outcomes with this preference average.

Definition A.1. Fix $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$. A *subjective mixture* $\alpha f \oplus (1-\alpha)g$ of f and g with weight α is any act $h \in \mathcal{F}$ such that $h(s) \sim \alpha f(s) \oplus (1-\alpha) g(s)$ for each $s \in S$.

As GM note, all subjective mixtures of f and g with weight α are statewise indifferent, and hence by Axiom 3 are indifferent. So we follow them and denote by $\alpha f \oplus (1-\alpha) g$ any one of them.

Finally we report GMMS's [11] result that the triple $(X, \sim, \oplus)$ constitutes a *mixture set*. Recall that $\alpha x \oplus (1-\alpha) y$ denotes the indifference class $\{z \in X | \ u(z) = \alpha u(x) + (1-\alpha)u(y)\}$. The triple $(X, \sim, \oplus)$ is a *mixture set* for equivalence classes if, for all $x, y \in X$ and all α, β in $[0, 1]$,

(M0) $\alpha x \oplus (1-\alpha) y \subset X$,
(M1) $x \in (1x \oplus 0y)$,
(M2) $\alpha x \oplus (1-\alpha) y = (1-\alpha) y \oplus \alpha x$ (commutative law),
(M3) $\beta(\alpha x \oplus (1-\alpha) y) \oplus (1-\beta) y = \alpha\beta x \oplus (1-\alpha\beta) y$ (distributive law).

Applying this result, state by state, to Definition A.1 (the definition of a subjective mixture of f and g with weight α in [0, 1]), it readily follows that the triple $(\mathcal{F}, \sim, \oplus)$ is also a *mixture set* and hence exhibits the analogous properties.

Lemma A.2 extends Axiom 5 to general subjective mixtures $\alpha f \oplus (1-\alpha) g$ of acts.

Lemma A.2. *Fix a preference order $\succsim$ satisfying Axioms* A0–A5. *For any $f, g, h \in \mathcal{F}$ such that $g, h \in \underline{\mathcal{F}}(f) \cap \overline{\mathcal{F}}(f)$ and $f \sim g$ hold, $\alpha g \oplus (1-\alpha) h \sim \alpha f \oplus (1-\alpha) h$ for all $\alpha \in [0, 1]$.*

Proof. Axioms A0–A4 imply the existence of a subjective mixture $\alpha f \oplus (1-\alpha) g$ as defined in Definition A.1. Note that for any three acts h, h', h'' satisfying $h', h'' \in \underline{\mathcal{F}}(h)$ we can conclude $\frac{1}{2}h' \oplus \frac{1}{2}h'' \in \underline{\mathcal{F}}(h)$. Similarly, for any three acts h, h', h'' with $h', h'' \in \overline{\mathcal{F}}(h)$ we can conclude $\frac{1}{2}h' \oplus \frac{1}{2}h'' \in \overline{\mathcal{F}}(h)$. Hence, we can appeal to the procedure of iterated averages in GMMS [11] in order to conclude that $h, h', h'' \in \mathcal{F}$ with $h', h'' \in \underline{\mathcal{F}}(h)$ implies $\alpha h' \oplus (1-\alpha)h'' \in \underline{\mathcal{F}}(h)$ for all $\alpha \in [0, 1]$. Similarly, $h, h', h'' \in \mathcal{F}$ with $h', h'' \in \overline{\mathcal{F}}(h)$ implies $\alpha h' \oplus (1-\alpha)h'' \in \overline{\mathcal{F}}(h)$ for all $\alpha \in [0, 1]$.

From Axiom 5.1 we know that $\frac{1}{2}g \oplus \frac{1}{2}h \succsim \frac{1}{2}f \oplus \frac{1}{2}h$ for $h \in \underline{\mathcal{F}}(f)$ and, from Axiom 5.2, $\frac{1}{2}f \oplus \frac{1}{2}h \succsim \frac{1}{2}g \oplus \frac{1}{2}h$ follows for $h \in \overline{\mathcal{F}}(f)$. Hence,

$$\tfrac{1}{2}f \oplus \tfrac{1}{2}h \sim \tfrac{1}{2}g \oplus \tfrac{1}{2}h$$

for all $g, h \in \underline{\mathcal{F}}(f) \cap \overline{\mathcal{F}}(f)$. Thus, we can use the procedure of iterated averages in GMMS [11] and conclude that $\alpha g \oplus (1-\alpha)h \sim \alpha f \oplus (1-\alpha)h$ holds for all $\alpha \in [0, 1]$. $\square$

Axiom 5 (Extreme events sensitivity) and Lemma A.2 imply that if the preference relation expresses indifference between two comonotonic acts then indifference is preserved when those two acts are each mixed with a third act that is pairwise comonotonic with both.

Lemma A.3 (*Comonotonic independence of indifference*). *Axiom* 5 *implies that $\succsim$ satisfies the following independence property for pairwise comonotonic acts. For any $\alpha \in [0, 1]$ and any three pairwise comonotonic acts $f, g, h \in \mathcal{F}$, if $f \sim g$ then $\alpha g \oplus (1-\alpha) h \sim \alpha f \oplus (1-\alpha) h$.*

Proof. From the pairwise comonotonicity of g and h with f, it follows that $g, h \in \underline{\mathcal{F}}(f) \cap \overline{\mathcal{F}}(f)$. Hence Lemma A.2 implies $\alpha f \oplus (1-\alpha) h \sim \alpha g \oplus (1-\alpha) h$ as required. $\square$

Proof of Theorem 5.1

1. *Sufficiency*: In Part (i) we first show that $\succsim$ has a CEU representation and in Part (ii) that the capacity is neo-additive.

Part (i): $\succsim$ *admits a CEU-representation*.

By Proposition 5.1 we know that Axioms A0–A4 imply that the preference relation $\succsim$ admits canonical biseparable representation,

$$V(x_E y) = v(E)\, u(x) + (1 - v(E))\, u(y).$$

The function $u(.)$ represents $\succsim$ restricted to the constant acts, and $V(x_E y) = v(E)\, u(x) + (1 - v(E))\, u(y)$ represents $\succsim$ restricted to the set of acts that are measurable with respect to a two-element partition of S.

Fix $f = [x_1 \text{ on } E_1; \ldots; \ x_n \text{ on } E_n]$. For each $i = 1, \ldots, n$, it follows from the definition of $\oplus$ and the connectedness of X, that there exists

- a unique $\lambda_i \in [0, 1]$ for which $x_i \in \lambda_i M \oplus (1 - \lambda_i)\, 0$ and
- a unique v_i for which

$$\left[M \text{ on } E_1 \cup \ldots \cup \ E_i; \ 0 \text{ on } E_{i+1} \cup \ldots \cup \ E_n\right] \sim v_i M \oplus (1 - v_i)\, 0.$$

Eq. (A.1) implies that $1 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ and $0 \leqslant v_1 \leqslant \cdots \leqslant v_{n-1} \leqslant 1$.

Hence we have, by construction and the mixture set properties of $(\mathcal{F}, \sim, \oplus)$ the following equalities:

$$f = \begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix} = \begin{bmatrix} \lambda_1 M \oplus (1 - \lambda_1)\, 0 & \text{on } E_1 \\ \vdots & \vdots \\ \lambda_n M \oplus (1 - \lambda_n)\, 0 & \text{on } E_n \end{bmatrix}$$
$$= (1 - \lambda_1) \begin{bmatrix} 0 & \text{on } E_1 \\ 0 & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix} \oplus (\lambda_1 - \lambda_2) \begin{bmatrix} M & \text{on } E_1 \\ 0 & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix}$$
$$\oplus\ (\lambda_2 - \lambda_3) \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix}$$
$$\oplus \cdots \oplus (\lambda_{n-1} - \lambda_n) \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ M & \text{on } E_3 \\ \vdots & \vdots \\ M & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix} \oplus \lambda_n \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ M & \text{on } E_3 \\ \vdots & \vdots \\ M & \text{on } E_{n-1} \\ M & \text{on } E_n \end{bmatrix}.$$

By applying the *comonotonic independence of indifference* property of Lemma A.3 $(n-1)$-times and utilizing the distributive law of $(\mathcal{F}, \sim, \oplus)$, we obtain

$$\begin{aligned} f &\sim (1-\lambda_1)\,0 \oplus (\lambda_1 - \lambda_2)\,[v_1 M \oplus (1-v_1)\,0] \oplus (\lambda_2-\lambda_3)\,[v_2 M \oplus (1-v_2)\,0] \\ &\quad \oplus \cdots \oplus (\lambda_{n-1}-\lambda_n)\left[v_{n-1}M \oplus (1-v_{n-1})\,0\right] \oplus \lambda_n M \\ &= \left[\sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\,v_i + \lambda_n\right] M \oplus \left[(1-\lambda_n) - \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\,v_i\right] 0. \end{aligned}$$

Hence it follows from Eq. (A.1) that for any pair of acts

$$f = \begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix} \quad \text{and} \quad f' = \begin{bmatrix} x'_1 & \text{on } E'_1 \\ \vdots & \vdots \\ x'_{n'} & \text{on } E'_{n'} \end{bmatrix}$$

applying the above methods we have

$$f \succsim f' \quad \text{if and only if}$$
$$\left[\sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\,v_i + \lambda_n\right] \geqslant \left[\sum_{j=1}^{n'-1} \left(\lambda'_j - \lambda'_{j+1}\right) v'_i + \lambda'_n\right].$$

By construction, $u(0) = 0$, $u(M) = 1$, $u(x_i) = \lambda_i$, $v(\emptyset) := 0$, $v(S) := 1$ and $v\left(\bigcup_{j=1}^{i} E_i\right) = v_i$.

Thus we have established that $\succsim$ can be represented by the CEU functional

$$\text{CEU}\left(\begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix}\right) = \sum_{i=1}^{n-1} (u(x_i) - u(x_{i+1}))\, v\left(\bigcup_{j=1}^{i} E_j\right) + u(x_n).$$

Part (ii): *v satisfies the hypotheses of Proposition* 3.1.

(*A*) *v is congruent with* $\mathcal{N}$.

From Part (i) we know that $\succsim$ admits a CEU representation with a capacity v. Hence, from Axiom 6, it follows immediately that $v(E) = 0$ and $v(S \backslash E) = 1$ holds for all $E \in \mathcal{N}$.

(*B*) *v satisfies conditions* (ii), (d), (a), (b) *and* (c), *of Proposition* 3.1.

(d) *We prove that for any* $E \in \mathcal{E}^*$ *and any* $F \in \mathcal{N}$ *such that* $E \cap F = \emptyset$, $v(E \cup F) = v(E)$.

Suppose $F \in \mathcal{N}$ and $E \in \mathcal{E}^*$. Consider the acts $f := y_E x_F y$, $g := y$, and $h := x_E y_F y$. Since $F \in \mathcal{N}$, we have $f \sim g$. Moreover, $h \in \underline{\mathcal{F}}(g) \cap \overline{\mathcal{F}}(g)$ and $f \in \underline{\mathcal{F}}(g) \cap \overline{\mathcal{F}}(g)$, since $F \in \mathcal{N}$. Hence, by Axiom 5,

$$\tfrac{1}{2} \cdot f \oplus \tfrac{1}{2} \cdot h \sim \tfrac{1}{2} \cdot g \oplus \tfrac{1}{2} \cdot h,$$

i.e.,

$$\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right)_{E \cup F} y \sim \left(\tfrac{1}{2}x + \tfrac{1}{2}y\right)_E y$$

or

$$v(E \cup F) = v(E).$$

(a) *We prove that for any three events* $E, F, G \in \mathcal{E}^*$ *such that* $E \cap F = \emptyset = E \cap G$, $E \cup F \in \mathcal{E}^*$ *and* $E \cup G \in \mathcal{E}^*$

$$v(E \cup F) - v(F) = v(E \cup G) - v(G).$$

The existence of four pairwise disjoint elements of $\mathcal{E}^*$ holds by hypothesis. Hence, we can assume that there are three pairwise disjoint events $E, F, G \in \mathcal{E}^*$ such that $E \cup F \cup G \in \mathcal{E}^*$. Lemma A.4 contains the key argument.

Lemma A.4. *If there are* $E, F, G \in \mathcal{E}^*$ *such that* $E \cap F = E \cap G = F \cap G = \emptyset$ *and* $E \cup F \cup G \in \mathcal{E}^*$, *then*

$$v(E \cup F \cup G) - v(F \cup G) = v(E \cup G) - v(F) \tag{A.2}$$

Proof. Assume, without loss of generality, $v(E \cup F) \leqslant v(F \cup G)$ and let $\beta \in [0, 1]$ be such that $v(E \cup F) = \beta \cdot v(F \cup G)$. Consider

$$f := \begin{bmatrix} M & \text{on} & E \cup F \\ 0 & \text{on} & S \backslash (E \cup F) \end{bmatrix},$$

$$f' := \begin{bmatrix} \beta \cdot M \oplus (1-\beta) \cdot 0 & \text{on} & F \cup G \\ 0 & \text{on} & S \backslash (F \cup G) \end{bmatrix},$$

$$f'' := \begin{bmatrix} M & \text{on} & F \cup G \\ 0 & \text{on} & S \backslash (F \cup G) \end{bmatrix}.$$

Clearly, $f \sim f'$, $f'' \in \underline{\mathcal{F}}(f) \cap \overline{\mathcal{F}}(f)$ and $f'' \in \underline{\mathcal{F}}(f') \cap \overline{\mathcal{F}}(f')$. By Axiom 5, we have

$$\tfrac{1}{2} \cdot f' \oplus \tfrac{1}{2} \cdot f'' \sim \tfrac{1}{2} \cdot f \oplus \tfrac{1}{2} \cdot f''.$$

Hence,

$$\begin{aligned} \tfrac{1}{2} \cdot [v(E \cup F \cup G) + v(F)] &= \tfrac{1}{2} \cdot (1+\beta) \cdot v(F \cup G) \\ &= \tfrac{1}{2} \cdot (v(F \cup G) + v(E \cup F)). \end{aligned}$$

Thus, we conclude $v(E \cup F \cup G) - v(F \cup G) = v(E \cup F) - v(F)$. $\square$

Let us now show that $(E, F, G) \in \mathcal{E}^*$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \in \mathcal{E}^*$ and $E \cup G \in \mathcal{E}^*$ implies

$$v(E \cup F) - v(F) = v(E \cup G) - v(G).$$

Several cases have to be considered when $F \neq G$.

Case 1.1: $F \subset G$. We have

$$\begin{aligned} v(E \cup G) - v(G) &= v(E \cup F \cup (G \backslash F)) - v(F \cup (G \backslash F)) \\ &= v(E \cup F) - v(F). \end{aligned}$$

The second equation follows from (d) for $G \backslash F \in \mathcal{N}$ and from Eq. (A.2) for $G \backslash F \in \mathcal{E}^*$.

Case 1.2: $G \subset F$. Similar to Case 1.1.

Case 2: $F \backslash G \neq \emptyset \neq G \backslash F$.

Case 2.1: $F \cap G \in \mathcal{E}^*$.

Either $F\backslash G \in \mathcal{N}$ or $F\backslash G \in \mathcal{E}^*$. Applying (d) or Eq. (A.2), we get

$$\begin{aligned} v(E \cup F) - v(F) &= v(E \cup (F \cap G) \cup (F\backslash G)) - v((F \cap G) \cup (F\backslash G)) \\ &= v(E \cup (F \cap G)) - v(F \cap G). \end{aligned}$$

Either $G\backslash F \in \mathcal{N}$ or $G\backslash F \in \mathcal{E}^*$. Applying (d) or Eq. (A.2), we get

$$\begin{aligned} v(E \cup (F \cap G)) - v(F \cap G) &= v(E \cup (F \cap G) \cup (G\backslash F)) - v((F \cap G) \cup (G\backslash F)) \\ &= v(E \cup G) - v(G). \end{aligned}$$

Hence, $v(E \cup F) - v(F) = v(E \cup G) - v(G)$.

Case 2.2: $F \cap G \in \mathcal{N}$.

From $F, G \in \mathcal{E}^*$ we can conclude $F\backslash G \in \mathcal{E}^*$ and $G\backslash F \in \mathcal{E}^*$.

Case 2.2.1: If $E \cup (F\backslash G) \cup (G\backslash F) \in \mathcal{E}^*$, then Eq. (A.2) implies

$$\begin{aligned} v(E \cup (F\backslash G)) - v(F\backslash G) &= v(E \cup (F\backslash G) \cup (G\backslash F)) - v((F\backslash G) \cup (G\backslash F)) \\ &= v(E \cup (G\backslash F)) - v(G\backslash F) \end{aligned}$$

and, applying (d), one has $v(E \cup F) - v(F) = v(E \cup G) - v(G)$.

Case 2.2.2: Let $E \cup (F\backslash G) \cup (G\backslash F) \in \mathcal{U}$ and consider a partition of S into four essential events $H_i \in \mathcal{E}^*$, $i = 1, 2, 3, 4$, whose existence is assumed by the hypothesis of the theorem. From $H_i \cap S = H_i$, for all $i = 1, 2, 3, 4$, it follows that, for each i, at least one of the sets $E \cap H_i$, $(F\backslash G) \cap H_i$ and $(G\backslash F) \cap H_i$ is in $\mathcal{E}^*$. Hence, by Property 4, this essential event can be partitioned into two essential events.

Suppose $E_1, E_2 \in \mathcal{E}^*$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. Applying Eq. (A.2) several times, we get

$$\begin{aligned} &v(E \cup (F\backslash G)) - v(F\backslash G) \\ &\quad = v(E_1 \cup E_2 \cup (F\backslash G)) - v(F\backslash G) \\ &\quad = [v(E_1 \cup E_2 \cup (F\backslash G)) - v(E_1 \cup (F\backslash G))] + [v(E_1 \cup (F\backslash G)) - v(F\backslash G)] \\ &\quad = [v(E_2 \cup (F\backslash G)) - v(F\backslash G)] + [v(E_1 \cup (F\backslash G)) - v(F\backslash G)] \\ &\quad = [v(E_2 \cup (G\backslash F)) - v(G\backslash F)] + [v(E_1 \cup (G\backslash F) - v(G\backslash F)] \\ &\quad = [v(E_1 \cup E_2 \cup (G\backslash F) - v(E_1 \cup (G\backslash F))] + [v(E_1 \cup (G\backslash F)) - v(G\backslash F)] \\ &\quad = v(E_1 \cup E_2 \cup (G\backslash F)) - v(G\backslash F) \\ &\quad = v(E \cup (G\backslash F)) - v(G\backslash F). \end{aligned}$$

Hence, from (d), we can conclude $v(E \cup F) - v(F) = v(E \cup G) - v(G)$. Similar reasoning applies, if $F\backslash G$ or $G\backslash F$ are the essential events which will be partitioned.

(b) *We prove that for some $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$,*

$$v(E \cup F) \leqslant v(E) + v(F).$$

Consider $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$. Assume, without loss of generality, that $v(E) = \beta \cdot v(F)$ for some $\beta \in [0, 1]$. Let

$$f := \begin{bmatrix} M & \text{on} & E \\ 0 & \text{on} & S\backslash E \end{bmatrix}, \quad f' := \begin{bmatrix} \beta \cdot M \oplus (1-\beta) \cdot 0 & \text{on} & F \\ 0 & \text{on} & S\backslash F \end{bmatrix},$$
$$f'' := \begin{bmatrix} M & \text{on} & F \\ 0 & \text{on} & S\backslash F \end{bmatrix}.$$

Clearly, $f \sim f'$, $f'' \in \underline{\mathcal{F}}(f)$ and $f'' \in \underline{\mathcal{F}}(f') \cap \overline{\mathcal{F}}(f')$. By Axiom 5(1), we have

$$\tfrac{1}{2} \cdot f' \oplus \tfrac{1}{2} \cdot f'' \succcurlyeq \tfrac{1}{2} \cdot f \oplus \tfrac{1}{2} \cdot f''.$$

Hence,

$$\begin{aligned}\tfrac{1}{2}\cdot v(E\cup F) &\leqslant \tfrac{1}{2}\cdot(1+\beta)\cdot v(F)\\ &= \tfrac{1}{2}\cdot(v(F)+v(E))\,.\end{aligned}$$

Thus, we conclude $v(E\cup F)\leqslant v(E)+v(F)$.

(c) *We prove that for some* $E, F\in\mathcal{E}^*$ *such that* $E\cap F=\emptyset$ *and* $E\cup F\notin\mathcal{U}$,

$$\overline{v}(E\cup F)\leqslant\overline{v}(E)+\overline{v}(F).$$

Consider $E, F\in\mathcal{E}^*$ such that $E\cap F=\emptyset$ and $E\cup F\notin\mathcal{U}$. Assume, without loss of generality, that $v(S\backslash E)=\beta\cdot v(S\backslash F)$ for some $\beta\in[0,1]$. Let

$$f:=\begin{bmatrix} M & \text{on} & S\backslash E\\ 0 & \text{on} & E\end{bmatrix},\quad f':=\begin{bmatrix}\beta\cdot M\oplus(1-\beta)\cdot 0 & \text{on} & S\backslash F\\ 0 & \text{on} & F\end{bmatrix},$$
$$f'':=\begin{bmatrix} M \text{ on } S\backslash F\\ 0\ \text{ on } F\end{bmatrix}.$$

Clearly, $f\sim f'$, $f''\in\overline{\mathcal{F}}(f)$ and $f''\in\underline{\mathcal{F}}(f')\cap\overline{\mathcal{F}}(f')$. By Axiom 5(2), we have

$$\tfrac{1}{2}\cdot f\oplus\tfrac{1}{2}\cdot f''\succsim\tfrac{1}{2}\cdot f'\oplus\tfrac{1}{2}\cdot f''.$$

Hence,

$$\begin{aligned}\tfrac{1}{2}+\tfrac{1}{2}\cdot v((S\backslash E)\cup(S\backslash F)) &\geqslant \tfrac{1}{2}\cdot(1+\beta)\cdot v(S\backslash F)\\ &= \tfrac{1}{2}\cdot(v(S\backslash F)+v(S\backslash E))\end{aligned}$$

or

$$\tfrac{1}{2}\cdot[(1-v(S\backslash F))+(1-v(S\backslash E))]\geqslant\tfrac{1}{2}\cdot[1-v((S\backslash E)\cup(S\backslash F))]\,.$$

Thus, we conclude $\overline{v}(E\cup F)\leqslant\overline{v}(E)+\overline{v}(F)$.

2. *Necessity*: The necessity of the representation follows straightforwardly from the definition of the neo-additive representation and so the proof is omitted. □

A.6. Remark on the minimal number of states

In order to see that four events are necessary for v to satisfy statement (ii a), consider the state space $S=\{s_1,s_2,s_3\}$ with $\mathcal{E}=2^S$ and $\mathcal{N}=\{\emptyset\}$. Let $\mathcal{F}$ be the set of acts $f: S\to\mathbb{R}$ with the preference relation $\succsim$ defined by $f\succsim g\iff\int f\,dv\geqslant\int g\,dv$ where the capacity v is defined as

$$\begin{aligned}&v(s_1)=\tfrac{1}{2},\quad v(s_2)=\tfrac{1}{3},\quad v(s_3)=\tfrac{1}{4},\\ &v(\{s_1,s_2\})=v(\{s_1,s_3\})=v(\{s_2,s_3\})=\tfrac{1}{2}.\end{aligned}$$

Note that v satisfies statements (ii b) and (ii c) but not (ii a) of Proposition 3.1. Since $\mathcal{E}$ contains three non-intersecting essential events, Proposition 3.1 implies that v is not neo-additive. Hence, three non-intersecting essential events do not suffice for the result of Theorem 5.1, because the preference order $\succsim$ induced by the capacity v satisfies all the axioms of the first part of Theorem 5.1. Except for *Extreme Events Sensitivity* (Axiom 5), it is not difficult to check this claim.

To see that Axiom 5 also obtains, note first that, by Eq. (5), for any two acts $f_1, f_2\in\mathcal{F}$,

$$\tfrac{1}{2}f_1\oplus\tfrac{1}{2}f_2=\tfrac{1}{2}[f_1+f_2]$$

holds. So, we have $\int\left(\frac{1}{2}f_1\oplus\frac{1}{2}f_2\right)dv=\int\frac{1}{2}(f_1+f_2)\,dv=\frac{1}{2}\int(f_1+f_2)\,dv$.

Moreover, for any two acts $f_1, f_2 \in \mathcal{F}$, such that $f_1 \in \underline{\mathcal{F}}(f_2)$, we have

$$\int f_1 \, dv + \int f_2 \, dv \geqslant \int (f_1 + f_2) \, dv. \tag{A.3}$$

To see this, assume without loss of generality, that $f_1(s_1) \leqslant f_1(s_2) \leqslant f_1(s_3)$ is true for act f_1. If $f_2(s_1) \leqslant f_2(s_2) \leqslant f_2(s_3)$ holds for the act f_2, then the claim follows from comonotonic additivity. Since $f_1 \in \underline{\mathcal{F}}(f_2)$, the only other case to consider is $f_2(s_1) \leqslant f_2(s_3) \leqslant f_2(s_2)$. Two subcases need to be checked. For notational convenience, we will write $f_{ij} := f_i(s_j)$, $v_i := v(s_i)$ and $v_{ij} := v(\{s_i, s_j\})$.

(i) Suppose $f_1(s_1) + f_2(s_1) \leqslant f_1(s_2) + f_2(s_2) \leqslant f_1(s_3) + f_2(s_3)$, then we have

$$\begin{aligned}
&\int f_1 \, dv + \int f_2 \, dv - \int (f_1 + f_2) \, dv \\
&\quad = f_{13} \cdot v_3 + f_{12} \cdot (v_{23} - v_3) + f_{11} \cdot (1 - v_{23}) \\
&\qquad + f_{22} \cdot v_2 + f_{23} \cdot (v_{23} - v_2) + f_{21} \cdot (1 - v_{23}) \\
&\qquad - [(f_{13} + f_{23}) \cdot v_3 + (f_{12} + f_{22}) \cdot (v_{23} - v_3) + (f_{11} + f_{21}) \cdot (1 - v_{23})] \\
&\quad = f_{22} \cdot v_2 + f_{23} \cdot (v_{23} - v_2) - [f_{23} \cdot v_3 + f_{22} \cdot (v_{23} - v_3)] \\
&\quad = (f_{22} - f_{23}) \cdot (v_2 + v_3 - v_{23}) = (f_{22} - f_{23}) \cdot \frac{1}{12} \geqslant 0.
\end{aligned}$$

(ii) Suppose $f_1(s_1) + f_2(s_1) \leqslant f_1(s_3) + f_2(s_3) \leqslant f_1(s_2) + f_2(s_2)$, then we have

$$\begin{aligned}
&\int f_1 \, dv + \int f_2 \, dv - \int (f_1 + f_2) \, dv \\
&\quad = f_{13} \cdot v_3 + f_{12} \cdot (v_{23} - v_3) + f_{11} \cdot (1 - v_{23}) \\
&\qquad + f_{22} \cdot v_2 + f_{23} \cdot (v_{23} - v_2) + f_{21} \cdot (1 - v_{23}) \\
&\qquad - [(f_{12} + f_{22}) \cdot v_2 + (f_{13} + f_{23}) \cdot (v_{23} - v_2) + (f_{11} + f_{21}) \cdot (1 - v_{23})] \\
&\quad = f_{13} \cdot v_3 + f_{12} \cdot (v_{23} - v_3) - [f_{12} \cdot v_2 + f_{13} \cdot (v_{23} - v_2)] \\
&\quad = (f_{13} - f_{12}) \cdot (v_2 + v_3 - v_{23}) = (f_{13} - f_{12}) \cdot \frac{1}{12} \geqslant 0.
\end{aligned}$$

Hence, Eq. (A.3) is true. An analogous computation shows that, for any two acts $f_1, f_2 \in \mathcal{F}$, such that $f_1 \in \overline{\mathcal{F}}(f_2)$,

$$\int f_1 \, dv + \int f_2 \, dv \leqslant \int (f_1 + f_2) \, dv \tag{A.4}$$

follows.

Consider now three acts $f, g, h \in \mathcal{F}$ such that $f \sim g$ and $h \in \underline{\mathcal{F}}(g) \cap \overline{\mathcal{F}}(g)$. Hence, $\int f \, dv = \int g \, dv$ holds. Moreover, by Eqs. (A.3) and (A.4), we get

$\int \left(\frac{1}{2} g \oplus \frac{1}{2} h\right) dv = \frac{1}{2} \int (g + h) \, dv = \frac{1}{2} \left[\int g \, dv + \int h \, dv\right] = \frac{1}{2} \left[\int f \, dv + \int h \, dv\right]$, where the last equality follows from the premise $f \sim g$.

Axiom 5.1 is satisfied, since by Eq. (A.3) $h \in \underline{\mathcal{F}}(f)$ implies $\int \left(\frac{1}{2} g \oplus \frac{1}{2} h\right) dv = \frac{1}{2} \left[\int f \, dv + \int h \, dv\right] \geqslant \frac{1}{2} \int (f + h) \, dv = \int \left(\frac{1}{2} f \oplus \frac{1}{2} h\right) dv$.

Similarly, we can show that Axiom 5.2 holds, since $h \in \overline{\mathcal{F}}(f)$ implies $\int \left(\frac{1}{2} f \oplus \frac{1}{2} h\right) dv = \frac{1}{2} \int (f + h) \, dv \geqslant \frac{1}{2} \left[\int f \, dv + \int h \, dv\right] = \int \left(\frac{1}{2} g \oplus \frac{1}{2} h\right) dv$.

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