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Many Good Risks: An Interpretation of Multivariate Risk and Risk Aversion without the Independence Axiom*

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What can be inferred, without assuming the Independence axiom, about an agent's preferences over many-good lotteries from knowledge that the agent is income risk averse? We show that income risk aversion corresponds to an intuitive substitution property of the many-good lottery preferences that is, itself, equivalent to a standard definition of many-good risk aversion. We apply our approach to derive some well-known results directly from our interpretation of income risk aversion and neutrality. *Journal of Economic Literature* Classification Numbers: D80, D81.

1. INTRODUCTION

To say that agents are income risk averse is to describe a property of their preferences over money lotteries. In an earlier paper, we argued that agents' primitive preferences were not for money itself but for the goods which money can buy.[1] We showed that it is not necessary to assume the Independence axiom in order for consistent money lottery preferences to be induced, in a natural way, from underlying many-good lottery preferences via a money metric utility transformation.

In this paper, still without assuming Independence, we show that income risk aversion in the induced money lottery preferences can be interpreted as having been inherited from a substitution property of the underlying many-good lottery preferences.[2]

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[1] Grant, Kajii, and Polak [6].

[2] For other formalizations of many-good risk aversion see Kihlstrom and Mirman [9], Levhari, Paroush and Peleg [10], Paroush [11], Rothblum [12], Duncan [4], Fishburn and Vickson [5], Karni [7, 8], Ambarish and Kellberg [1], and Chew and Epstein [3].

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Section 2 establishes notation and states some assumptions on preferences that will be maintained throughout the paper. It is well known, in the Expected Utility framework, what income risk aversion implies about underlying many-good lottery preferences. Section 3 extends this for underlying preferences that do not satisfy Independence and shows that income risk aversion is induced by an intuitive substitution axiom. Section 4 relates this axiom to a standard extension to many-good lotteries of the Rothschild–Stiglitz [13] definition of increasing risk. In Section 5 we show that some well-known results follow directly from this interpretation of income risk aversion and neutrality.

2. The Basic Framework

Let $\mathbf{X}$ be a set of outcomes. We assume $\mathbf{X}$ is a compact metric space. The example of $\mathbf{X} = \prod_{i=1}^{n} \mathbf{X}_i$, where, for each i, $\mathbf{X}_i$ is a closed interval, is of special interest. In this case we refer to $\mathbf{X}$ as a multi-commodity space and to the outcomes as commodity bundles. Let $\mathscr{L}(\mathbf{X})$ denote the space of lotteries (probability distributions) over $\mathbf{X}$ endowed with the weak convergence topology. Let $\boldsymbol{\delta}_{\mathbf{x}}$ denote the degenerate lottery on outcome $\mathbf{x} \in \mathbf{X}$. For $\alpha \in [0, 1]$, let $\alpha\mathbf{F} \oplus (1-\alpha)\,\mathbf{G} \in \mathscr{L}(\mathbf{X})$ denote the probability mixture of $\mathbf{F}$ and $\mathbf{G}$.

We assume that the agent has preferences over $\mathscr{L}(\mathbf{X})$ denoted by $\succsim$. In the following analysis we take these preferences as our primitive and assume that they are reflexive, transitive, complete, continuous, and obey the *Reduction of Compound Lottery Axiom*. Let $\sim$ correspond to indifference and $\succ$ to strict preference. We can define the preference ordering restricted to a non-empty subset $\mathscr{Q}$ of $\mathscr{L}(\mathbf{X})$, denoted $\succsim_{\mathscr{Q}}$. With slight abuse of notation, we will write $\succsim_{\mathbf{X}}$ to mean the agent's preferences over outcomes.

Throughout this paper, we assume that $\succsim$ respect Fishburn and Vickson's [5] "non-dimensional first order stochastic dominance." That is, $\succsim$ obey the following simple substitution axiom:

Axiom of Degenerate Independence (*ADI*). For all $\mathbf{F} \in \mathscr{L}(\mathbf{X})$, for all $\alpha \in (0, 1)$, and for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim_{\mathbf{X}} \mathbf{y} \text{ if and only if } (1-\alpha)\,\mathbf{F} \oplus \alpha\boldsymbol{\delta}_{\mathbf{x}} \succsim (1-\alpha)\,\mathbf{F} \oplus \alpha\boldsymbol{\delta}_{\mathbf{y}}.$$

We call $p(\cdot)$, a real-valued continuous function on $\mathbf{X}$, a *price system*. We regard $p(\mathbf{x})$ as the money cost of outcome $\mathbf{x}$. When $\mathbf{X}$ is a multi-commodity space we assume $p(\cdot)$ is a linear function, and (with slight abuse of notation) we denote $p(\mathbf{x}) = p \cdot \mathbf{x}$. In this case, we assume p is strictly positive. Let $\mathbf{Y}(p)$ be the *income expansion path* for a given price system p; that is,

$$\mathbf{Y}(p) = \{\mathbf{y} \in \mathbf{X} \colon \exists \bar{\mathbf{x}} \in \mathbf{X} \text{ s.t. } \mathbf{y} \in \underset{\langle \mathbf{x} \rangle}{\operatorname{argmin}}\, p(\mathbf{x}) \text{ s.t. } \mathbf{x} \succsim_{\mathbf{X}} \bar{\mathbf{x}}\} \subseteq \mathbf{X}.$$

Thus $\succsim_{\mathscr{L}(\mathbf{Y}(p))}$ refers to the restriction of $\succsim$ to lotteries over that income expansion path.

Define a mapping $m_p \colon \mathbf{X} \to \mathbb{R}$ such that

$$m_p(\mathbf{x}) \equiv \min_{\hat{\mathbf{x}}} p(\hat{\mathbf{x}}) \qquad \text{s.t. } \hat{\mathbf{x}} \succsim_{\mathbf{X}} \mathbf{x}.$$

Let $\mathbf{M}_p =$ range of m_p and let $\mathscr{L}(\mathbf{M}_p)$ be the space of lotteries over $\mathbf{M}_p$. We call $\mathbf{D} \in \mathscr{L}(\mathbf{M}_p)$ a money lottery. We can then define the *money metric utility transformation*,

$$T_p \colon \mathscr{L}(\mathbf{X}) \to \mathscr{L}(\mathbf{M}_p) \qquad \text{s.t.} \quad \forall \mathbf{F} \in \mathscr{L}(\mathbf{X})\ T_p(\mathbf{F}) = \mathbf{F} \circ m_p^{-1}.$$

Preferences over money lotteries, denoted $\succsim_{\mathscr{L}(\mathbf{M}_p)}$, are given by

$$\text{for all } \mathbf{F}, \mathbf{G} \in \mathscr{L}(\mathbf{X}), \qquad T_p(\mathbf{F}) \succsim_{\mathscr{L}(\mathbf{M}_p)} T_p(\mathbf{G}) \text{ if and only if } \mathbf{F} \succsim \mathbf{G}.$$

Grant, Kajii, and Polak [6, Proposition 3.2] show that $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ are well-defined and respect FSD if and only if $\succsim$ obey ADI.

3. Income Risk Aversion

In this section we ask what can be inferred, assuming ADI, about underlying preferences over many-good lotteries from knowledge that induced preferences over money lotteries exhibit income risk aversion, first for given prices, and then for all prices? Alternatively, what do we have to assume about the underlying preferences in order to induce income risk aversion for preferences over money lotteries?

We define risk for a univariate distribution (i.e., income risk) by the notion of Weak Second Order Stochastic Dominance (WSSD). Therefore our definition of income risk aversion is as follows:

Definition (Income Risk Aversion)[3]. We say $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ exhibit *income risk aversion* if for all $\mathbf{D}, \mathbf{E} \in \mathscr{L}(\mathbf{M}_p)$,

$$\int_{s=0}^{m} \mathbf{D}(s)\, ds \leqslant \int_{s=0}^{m} \mathbf{E}(s)\, ds \text{ for all } m \text{ implies } \mathbf{D} \succsim_{\mathscr{L}(\mathbf{M}_p)} \mathbf{E}.$$

[3] Note that this definition includes risk neutrality as a limiting case. WSSD is similar to the notion of "increasing risk" used by Rothschild and Stiglitz [13], except that "increasing risk" only compares distributions of the same mean.

If $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ are those preferences over money lotteries induced from underlying preferences over many-good lotteries, $\succsim$, by the money metric transformation for a *particular* price system p, then $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ exhibit income risk aversion if and only if $\succsim$ satisfy the following substitution property:[4]

Axiom of Income Risk Aversion (*AIRA*). We say, $\succsim$ satisfy AIRA with respect to a price system p, if

$$\forall \mathbf{x}^p, \mathbf{z}^p, \mathbf{y}^p \in \mathbf{Y}(p) \qquad \text{s.t.} \quad \mathbf{x}^p \succ \mathbf{z}^p \succ \mathbf{y}^p$$

(i.e., for three points on the income expansion path defined for the price system p),

$$\forall \alpha \in [0, 1], \forall \mathbf{F} \in \mathscr{L}(\mathbf{Y}(p))$$

$$\alpha \mathbf{F} \oplus (1-\alpha)\, \boldsymbol{\delta}_{\mathbf{z}^p} \succsim \alpha \mathbf{F} \oplus (1-\alpha)[\beta \cdot \boldsymbol{\delta}_{\mathbf{x}^p} \oplus (1-\beta)\, \boldsymbol{\delta}_{\mathbf{y}^p}],$$

where $\beta \in (0, 1)$ is defined by $p(\mathbf{z}^p) = \beta p(\mathbf{x}^p) + (1-\beta)\, p(\mathbf{y}^p)$.

PROPOSITION 3.1. *If $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ are induced from $\succsim$, by $T_p(\mathbf{F}) = \mathbf{F} \circ m_p^{-1}$, then $\succsim$ satisfy* AIRA *for the price system p iff $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ exhibit income risk aversion.*

Proof. See Appendix.

AIRA accords with the intuition that mean-income preserving splits of probability masses cannot be welfare improving for an income risk averter. Note that AIRA, like ADI, is a substitution property that does not rely on $\mathbf{X}$ being a vector space. ADI only uses the ordering of outcomes implied by $\succsim_{\mathbf{X}}$, while AIRA uses the linear structure of the labels assigned to indifference sets in $\mathbf{X}$ by the money metric utility representation of $\succsim_{\mathbf{X}}$. However, in what follows we shall restrict our attention to $\mathbf{X}$ being a multi-commodity space. We will assume, further, that preferences over commodity bundles, $\succsim_{\mathbf{X}}$, are strictly monotonic.

Having determined what property of underlying preferences induces (and is implied by) income risk aversion for *one* price vector, the next question is what property induces (and is implied by) income risk aversion for *all* positive price vectors. One partial answer is the following substitution axiom:

Axiom of Bifurcation Risk Aversion (*ABRA*). We say that $\succsim$ obey ABRA, if and only if,

$$\forall \mathbf{F} \in \mathscr{L}(\mathbf{X}), \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}, \forall \alpha, \beta \in [0, 1]$$

$$\alpha \mathbf{F} \oplus (1-\alpha)\, \boldsymbol{\delta}_{(\beta \mathbf{x}_1 + [1-\beta]\mathbf{x}_2)} \succsim \alpha \mathbf{F} \oplus (1-\alpha)[\beta \boldsymbol{\delta}_{\mathbf{x}_1} \oplus (1-\beta)\, \boldsymbol{\delta}_{\mathbf{x}_2}].$$

[4] Recall that we are assuming $\succsim$ satisfy ADI and hence induced money lottery preferences $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ exhibit FSD.

PROPOSITION 3.2. *Given* $\succsim_X$ *strictly monotonic,* $\succsim$ *satisfy* ABRA *iff* $\succsim$ *satisfy* AIRA *for all* $p \in \mathbb{R}^n_{++}$ *and* $\succsim_X$ *convex.*

Proof. "$\Rightarrow$" (i) ABRA $\Rightarrow$ $\succsim_X$ convex. If $\succsim_X$ are non-convex, then we can find $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ such that, for some $\alpha \in (0, 1)$, $\alpha\mathbf{x} + (1-\alpha)\,\mathbf{y} = \mathbf{z}$ and such that $\mathbf{x} \succsim_X \mathbf{y} \succ_X \mathbf{z}$. Hence, by ADI, $\alpha\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\alpha)\,\boldsymbol{\delta}_{\mathbf{y}} \succ \boldsymbol{\delta}_{\mathbf{z}}$ which violates ABRA.

(ii) ABRA $\Rightarrow$ AIRA $\forall p \in \mathbb{R}^n_{++}$ (see Fig. 1a). For a particular

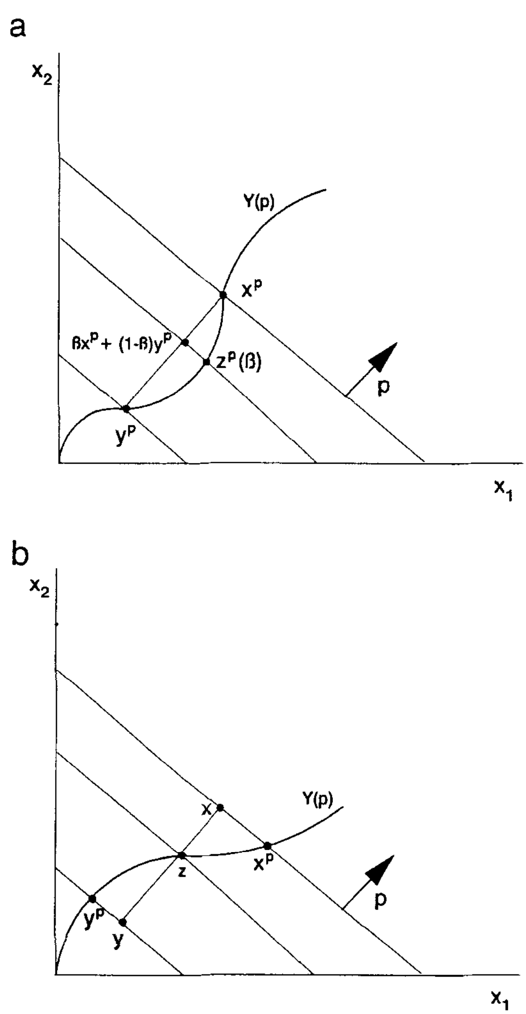

FIG. 1. An illustration, given the convexity of $\succsim_X$, of the equivalence of ABRA and AIRA for all positive price vectors in the case of two goods.

$p \in \mathbb{R}^n_{++}$, consider 2 points $\mathbf{x}^p, \mathbf{y}^p \in \mathbf{Y}(p)$ s.t. $\mathbf{x}^p \succ \mathbf{y}^p$. By ABRA, $\forall \mathbf{F} \in \mathscr{L}(\mathbf{X})$, $\forall \alpha, \beta \in [0, 1]$,

$$\alpha\mathbf{F} \oplus (1-\alpha)\,\boldsymbol{\delta}_{(\beta\mathbf{x}^p + (1-\beta)\mathbf{y}^p)} \succsim \alpha\mathbf{F} \oplus (1-\alpha)[\beta\boldsymbol{\delta}_{\mathbf{x}^p} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}^p}]$$

and by revealed preference and ADI,

$$\alpha\mathbf{F} \oplus (1-\alpha)\,\boldsymbol{\delta}_{\mathbf{z}^p(\beta)} \succsim \alpha\mathbf{F} \oplus (1-\alpha)\,\boldsymbol{\delta}_{(\beta\mathbf{x}^p + (1-\beta)\mathbf{y}^p)},$$

where $\mathbf{z}^p(\beta) \in \mathbf{Y}(p)$ and $p \cdot \mathbf{z}^p(\beta) = p \cdot [\beta\mathbf{x}^p + (1-\beta)\,\mathbf{y}^p]$. ∎

"$\Leftarrow$" (See Fig. 1b). Consider any $\mathbf{z} \in \mathbf{X}$. Since $\succsim_{\mathbf{X}}$ are convex and strictly monotonic, there exists a $p \in \mathbb{R}^n_{++}$ s.t. $\mathbf{z} \in \mathbf{Y}(p)$. Note that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ s.t. $\beta\mathbf{x} + (1-\beta)\,\mathbf{y} = \mathbf{z}$ for some $\beta \in [0, 1]$, there exist $\mathbf{x}^p, \mathbf{y}^p \in \mathbf{Y}(p)$ s.t. $p \cdot \mathbf{x}^p = p \cdot \mathbf{x}$ and $p \cdot \mathbf{y}^p = p \cdot \mathbf{y}$. By ADI, $\forall \mathbf{F} \in \mathscr{L}(\mathbf{X})$, $\exists \mathbf{F}' \in \mathscr{L}(\mathbf{Y}(p))$ s.t. $\forall \mathbf{G} \in \mathscr{L}(\mathbf{X})$ and $\forall \alpha \in [0, 1]$, $\alpha\mathbf{F}' \oplus (1-\alpha)\mathbf{G} \sim \alpha\mathbf{F} \oplus (1-\alpha)\,\mathbf{G}$. Applying AIRA for prices p we have $\alpha\mathbf{F}' \oplus (1-\alpha)\,\boldsymbol{\delta}_{\mathbf{z}} \succsim \alpha\mathbf{F}' \oplus (1-\alpha)[\beta\boldsymbol{\delta}_{\mathbf{x}^p} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}^p}]$. Since $\mathbf{z} = \beta\mathbf{x} + (1-\beta)\,\mathbf{y}$ and $\mathbf{x}^p \succsim_{\mathbf{X}} \mathbf{x}$, $\mathbf{y}^p \succsim_{\mathbf{X}} \mathbf{y}$ (by revealed preference), applying ADI twice we have:

$$\alpha\mathbf{F}' \oplus (1-\alpha)\,\boldsymbol{\delta}_{(\beta\mathbf{x} + (1-\beta)\mathbf{y})} \succsim \alpha\mathbf{F}' \oplus (1-\alpha)[\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}]$$

and hence

$$\alpha\mathbf{F} \oplus (1-\alpha)\,\boldsymbol{\delta}_{(\beta\mathbf{x} + (1-\beta)\mathbf{y})} \succsim \alpha\mathbf{F} \oplus (1-\alpha)[\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}]. \quad ∎$$

An immediate consequence of Propositions 3.1 and 3.2 is that ABRA induces income risk aversion in the money lottery preferences induced by a money metric utility transformation defined for *any* positive price vector. Moreover, if we know that $\succsim_{\mathbf{X}}$ are strictly convex and that the agent is income risk averse for all prices, then we can infer that $\succsim$ satisfy ABRA, whether or not they satisfy Independence.

4. ABRA AND MANY-GOOD RISK AVERSION

Given that ABRA is almost equivalent to income risk aversion for all prices, how does ABRA compare with standard definitions of many-good risk aversion in the Expected Utility literature?

Note first that ABRA captures the intuition that a mean-preserving *linear bifurcation* of a probability mass in a distribution increases risk and hence makes a risk averse agent worse off. However, not all multi-dimensional mean-preserving spreads are ranked by ABRA.[5] For example,

[5] By $\mathbf{F}$ is a multi-dimentional mean-preserving spread of $\mathbf{G}$, we mean that for each coordinate, the marginal of $\mathbf{F}$ is a mean-preserving spread of the marginal of $\mathbf{G}$.

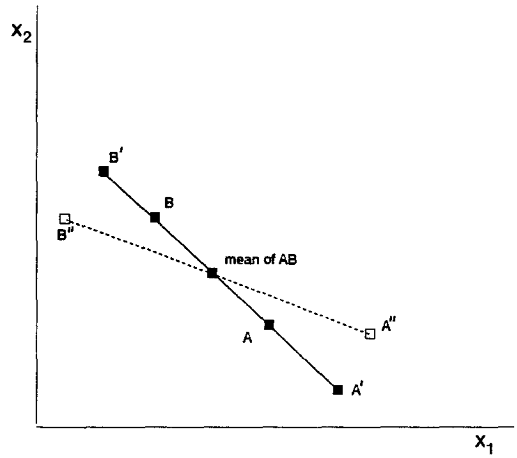

FIG. 2. Let each ■ and □ represent a probability mass of $\frac{1}{2}$. Hence let AB refer to the lottery $\frac{1}{2}\boldsymbol{\delta}_A \oplus \frac{1}{2}\boldsymbol{\delta}_B$, etc. Both lotteries $A'B'$ and $A''B''$ have the same mean outcome as AB, but only $A'B'$ can be obtained from AB by a sequence (of two) mean-preserving linear bifurcations.

consider the simple distribution in Fig. 2 which consists of two probability masses of $\frac{1}{2}$ on commodity bundles A and B. Let AB refer to the lottery $\frac{1}{2}\boldsymbol{\delta}_A \oplus \frac{1}{2}\boldsymbol{\delta}_B$, etc. It is easy to verify that the distribution $A'B'$ can be obtained as the result of two linear bifurcations on AB, and thus for preferences which satisfy ABRA we can deduce that $AB \succsim A'B'$. But although $A''B''$ can also be viewed as a mean-preserving spread of AB, it cannot be derived from AB as a sequence of linear bifurcations, and thus ABRA has no implication for the preference between these two distributions. Note that lottery $A''B''$ can be regarded as a "small perturbation" of $A'B'$. Hence, the example also illustrates that the relation induced by aversion to mean-preserving linear bifurcations is *not* continuous with respect to the natural topology, unlike its one-dimensional analog.

ABRA is, however, equivalent to a particular definition of multi-dimensional risk aversion. Recall that one interpretation of the Rothschild–Stiglitz [13] one-dimensional definition of "increasing risk" is the addition of noise to a random variable. Following Fishburn and Vickson [5], we can extend this definition of increasing risk to multi-commodity space.

DEFINITION (Many-Good Noisier [MGN]). For $\mathbf{F}, \mathbf{G} \in \mathscr{L}(\mathbf{X})$, we say $\mathbf{G}$ is *many-good noisier* (*MGN*) than $\mathbf{F}$ if and only if there exist random variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, where

(i) $\mathbf{F}$ (resp. $\mathbf{G}$) is the distribution of $\mathbf{X}$ (resp. $\mathbf{Y}$)

and

(ii) $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ with $\mathbf{E}[\mathbf{Z} \mid \mathbf{X} = \mathbf{x}] = 0$ for all $\mathbf{x} \in \text{range}(\mathbf{X})$.

DEFINITION (Many-good Noise Aversion [MGNA]). We say that $\succsim$ are *many-good noise averse* if and only if

$$\mathbf{G}\ \mathrm{MGN}\ \mathbf{F} \text{ implies } \mathbf{F} \succsim \mathbf{G}.$$

PROPOSITION 4.1. *Preferences over many-good lotteries,* $\succsim$, *satisfy* ABRA *if and only if* $\succsim$ *are many-good noise averse.*

For simple lotteries, if one lottery is noisier than the other, it is straightforward to derive the former from the latter by a sequence of linear mean-preserving bifurcations. Hence the proof follows immediately from transitivity of $\succsim$ and from the definition of ABRA. The proof for the case of continuous lotteries follows immediately (given continuity of $\succsim$) from the following lemma.[6].

LEMMA 4.2. *For all* $\mathbf{F}, \mathbf{G} \in \mathscr{L}(\mathbf{X})$, $\mathbf{G}\ \mathrm{MGN}\ \mathbf{F}$ *implies that there exist sequences of simple distributions,* $\langle \mathbf{G}_m \rangle$ *and* $\langle \mathbf{F}_m \rangle$, *such that* $\mathbf{G}_m \to \mathbf{G}$, $\mathbf{F}_m \to \mathbf{F}$, *and* $\mathbf{G}_m\ \mathrm{MGN}\ \mathbf{F}_m$ *for all m. Moreover,* $\mathbf{F}_m$ *and* $\mathbf{G}_m$ *can be taken such that mean of* $\mathbf{F}_m =$ *mean of* $\mathbf{G}_m =$ *mean of* $\mathbf{F}$ ($=$*mean of* $\mathbf{G}$).

Proof. See Appendix.

In the Expected Utility framework, income risk aversion for all prices has usually been defined as the agent's having a concave von Neumann–Morgernstern (vN–M) utility function for commodity bundles. Fishburn and Vickson [5] show that this is equivalent to MGNA. Without the Independence axiom, the concave vN–M utility function definition is not applicable, but the above argument shows that (given convex outcome preferences) MGNA is still equivalent to income risk aversion for all prices. The substitution axiom, ABRA, enables us to work with the notion of MGNA directly.

As ABRA is too weak to rank all multi-dimensional mean-preserving spreads, one might be tempted to strengthen it. It could be argued, however, that ABRA is capturing more than just risk aversion since, with ADI, it imposes convexity on $\succsim_{\mathbf{X}}$. It is not hard to imagine preferences that violate convexity in $\mathbf{X}$ but which exhibit "risk aversion" along, say, any ray from the origin. However, non-convex outcome preferences often have some income expansion paths that are disconnected, and thus it is not surprising that for these prices the individual's induced preferences over money lotteries might not exhibit global income risk aversion.[7] Income

[6] This result is probably well known in the literature but we include it for completeness.

[7] We are grateful to Professor Mas-Colell for bringing this point to our attention.

expansion paths for strictly convex preferences over commodity bundles are always connected and so this problem does not arise.[8].

5. Income and Bifurcation Risk Neutrality

Grant, Kajii, and Polak [6] showed that $\succsim_{\mathscr{L}(\mathbf{M})}$ respecting FSD restricts the underlying preferences over lotteries with regard to substitutions of a single point by another single point. Similarly, the previous sections show that $\succsim_{\mathscr{L}(\mathbf{M})}$ respecting WSSD, i.e., risk aversion, restricts $\succsim$ with regard to substitutions of a single point by particular two-point distributions. One nice aspect of this interpretation of income risk aversion is that it provides a simple and direct way to prove certain theorems about income risk neutrality without recourse to mathematical properties of any preference representation.

Let us specialize the definitions from above to incorporate income risk neutrality for one price vector and for all positive price vectors.

Income Risk Neutrality (*IRN*). We say, $\succsim$ satisfy IRN with respect to a price system p and a connected set $\mathbf{C}(p) \subseteq \mathbf{Y}(p)$, if,

$$\forall \mathbf{x}^p, \mathbf{z}^p, \mathbf{y}^p \in \mathbf{C}(p) \qquad \text{s.t.} \quad \mathbf{x}^p \succ \mathbf{z}^p \succ \mathbf{y}^p$$
$$\forall \alpha \in [0, 1], \forall \mathbf{F} \in \mathscr{L}(\mathbf{C}(p))$$
$$\alpha \mathbf{F} \oplus (1-\alpha)\, \boldsymbol{\delta}_{\mathbf{z}^p} \sim \alpha \mathbf{F} \oplus (1-\alpha)[\beta \cdot \boldsymbol{\delta}_{\mathbf{x}^p} \oplus (1-\beta)\, \boldsymbol{\delta}_{\mathbf{y}^p}],$$

where $\beta \in (0, 1)$ is defined by $p(\mathbf{z}^p) = \beta p(\mathbf{x}^p) + (1-\beta)\, p(\mathbf{y}^p)$.

Bifurcation Risk Neutrality (*BRN*). We say that $\succsim$ satisfy BRN if and only if,

$$\forall \mathbf{F} \in \mathscr{L}(\mathbf{X}), \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}, \forall \alpha, \beta \in [0, 1]$$
$$\alpha \mathbf{F} \oplus (1-\alpha)\, \boldsymbol{\delta}_{(\beta \mathbf{x}_1 + [1-\beta]\mathbf{x}_2)} \sim \alpha \mathbf{F} \oplus (1-\alpha)[\beta \boldsymbol{\delta}_{\mathbf{x}_1} \oplus (1-\beta)\, \boldsymbol{\delta}_{\mathbf{x}_2}].$$

Safra and Segal [14, Theorem 6] have shown that if preferences satisfy ADI and a notion of *many-good risk neutrality* that is weaker than BRN, then $\succsim$ satisfy Independence. In our framework we can show directly that, even if preferences are only *income risk neutral* along a segment of a single income expansion path (IEP), then the underlying preferences over many-good lotteries whose support lies within the indifference sets that corresponds to that segment obey Independence.

[8] The case of weakly convex preferences (i.e., indifference curves with flat portions) is tricky but the reader can readily verify that ABRA places enough restrictions on the relative slopes of the flat portions for Proposition 4.2 to go through.

PROPOSITION 5.1. *If* $\succsim$ *satisfy* IRN *with respect to some* p *and some* $\mathbf{C}(p) \subseteq \mathbf{Y}(p)$, *then* $\succsim_{\mathscr{L}(\mathbf{Z})}$ *satisfy Independence, where* $\mathbf{Z} \equiv \{\mathbf{z} \in \mathbf{X} \mid \exists \mathbf{y} \in \mathbf{C}(p), \mathbf{z} \sim_{\mathbf{X}} \mathbf{y}\}$.

Proof. For any $\mathbf{F}, \mathbf{G}, \mathbf{H} \in \mathscr{L}(\mathbf{Z})$, we can find $\mathbf{F}', \mathbf{G}', \mathbf{H}' \in \mathscr{L}(\mathbf{C}(p))$ such that $T_p(\mathbf{F}) = T_p(\mathbf{F}')$, $T_p(\mathbf{G}) = T_p(\mathbf{G}')$, $T_p(\mathbf{H}) = T_p(\mathbf{H}')$. Choose any $\alpha \in (0, 1]$. By ADI (and continuity of $\succsim$),

$$\alpha \mathbf{F} \oplus (1-\alpha)\mathbf{H} \sim \alpha \mathbf{F}' \oplus (1-\alpha)\mathbf{H}'$$

and

$$\alpha \mathbf{G} \oplus (1-\alpha)\mathbf{H} \sim \alpha \mathbf{G}' \oplus (1-\alpha)\mathbf{H}'.$$

Let $\mathbf{x}_{\mathbf{F}'}$, (resp. $\mathbf{x}_{\mathbf{G}'}$) $\in \mathbf{C}(p)$, where $p \cdot \mathbf{x}_{\mathbf{F}'} = \int p \cdot \mathbf{x} \, d\mathbf{F}'(\mathbf{x})$. By IRN,

$$\alpha \mathbf{F}' \oplus (1-\alpha)\mathbf{H}' \sim_{\mathscr{L}(\mathbf{Z})} \alpha \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{F}'}} \oplus (1-\alpha)\mathbf{H}'$$

and

$$\alpha \mathbf{G}' \oplus (1-\alpha)\mathbf{H}' \sim_{\mathscr{L}(\mathbf{Z})} \alpha \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{G}'}} \oplus (1-\alpha)\mathbf{H}'.$$

And, by ADI,

$$\alpha \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{F}'}} \oplus (1-\alpha)\mathbf{H}' \succsim_{\mathscr{L}(\mathbf{Z})} \alpha \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{G}'}} \oplus (1-\alpha)\mathbf{H}' \Leftrightarrow \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{F}'}} \succsim_{\mathscr{L}(\mathbf{Z})} \boldsymbol{\delta}_{\mathbf{x}_{\mathbf{G}'}}$$

Hence

$$\alpha \mathbf{F} \oplus (1-\alpha)\mathbf{H} \succsim \alpha \mathbf{G} \oplus (1-\alpha)\mathbf{H} \Leftrightarrow \mathbf{F} \succsim \mathbf{G}. \quad \blacksquare$$

Note that, if the segment, $\mathbf{C}(p)$, was the whole of an IEP, then $\succsim$ satisfy Independence for lotteries over the whole of $\mathbf{X}$.

In Proposition 5.1, $\succsim$ do not necessarily satisfy ABRA (or income risk aversion for all prices). Imposing ABRA leads to the next proposition. Stiglitz [15] pointed out that if an agent is income risk neutral along all IEPs then they must all be linear.[9] Our approach provides some intuition for Stiglitz's result. In fact we can slightly strengthen this result to show that income risk neutrality along any segment of an IEP implies the convexity (linearity, if outcome preferences are strictly convex) of that segment.

PROPOSITION 5.2. *Given* $\succsim_{\mathbf{X}}$ *convex (resp. strictly convex) and strictly monotonic, if* $\succsim$ *obeys* AIRA *with respect to all* $p \in \mathbb{R}^n_{++}$ *and if* $\succsim$ *satisfy*

[9] In fact, Stiglitz shows that preferences must be homothetic if any Arrow–Pratt coefficient of income risk aversion is to apply for all prices. We confine ourselves to the case of income risk, neutrality, since in general the Arrow–Pratt measure does not extend to non-Expected Utility theory.

IRN *with respect to some* q *and some segment* $\mathbf{C}(q) \subseteq \mathbf{Y}(q)$, *then* $\mathbf{C}(q)$ *is convex* (*resp. linear*).[10]

Proof (by contradiction). Suppose that the risk neutral segment of $\mathbf{Y}(q)$ is not convex; i.e., there exists $\mathbf{x}^q, \mathbf{y}^q \in \mathbf{Y}(q)$ s.t. $\mathbf{x}^q \succ_{\mathbf{X}} \mathbf{y}^q$ and $\mathbf{z} \equiv \frac{1}{2}\mathbf{x}^q + \frac{1}{2}\mathbf{y}^q \notin \mathbf{Y}(q)$. We can find $\mathbf{z}^q \in \mathbf{Y}(q)$ s.t. $q \cdot \mathbf{z}^q = q \cdot \mathbf{z}$. Note, by revealed preference, $\mathbf{z} \prec \mathbf{z}^q$. Then, by income risk neutrality for q, $\boldsymbol{\delta}_{\mathbf{z}^q} \sim \frac{1}{2}\boldsymbol{\delta}_{\mathbf{x}^q} \oplus \frac{1}{2}\boldsymbol{\delta}_{\mathbf{y}^q}$, hence by ADI, $\boldsymbol{\delta}_{\mathbf{z}} \prec \frac{1}{2}\boldsymbol{\delta}_{\mathbf{x}^q} \oplus \frac{1}{2}\boldsymbol{\delta}_{\mathbf{y}^q}$. This is a violation of ABRA, but we know, by Proposition 3.2, that ABRA obtains. Thus we have a contradiction. The same argument can be employed to show that with strictly convex preferences the segment of the IEP must be linear. ∎

The contrapositive of Proposition 5.2 illustrates the restrictiveness of ABRA; the agent cannot be risk neutral along the entirely of a non-linear segment of an income expansion path. Moreover, as the next proposition shows, if the individual exhibits risk aversion along all income expansion paths defined for positive prices, then a bifurcation involving commodity bundles that do not lie on the same income expansion path makes the individual *strictly* worse off.

PROPOSITION 5.3. *Given* $\succsim_{\mathbf{X}}$ *strictly monotonic, if* $\succsim$ *obey* ABRA, *then* $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$ s.t. $\mathbf{x}$ *and* $\mathbf{y}$ *do not lie on the same income expansion path* (IEP), $\forall \alpha, \beta \in (0, 1)$, *and* $\forall \mathbf{F} \in \mathscr{L}(\mathbf{X})$,

$$(1-\alpha)\,\mathbf{F} \oplus \alpha(\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}) \prec (1-\alpha)\,\mathbf{F} \oplus \alpha\boldsymbol{\delta}_{\mathbf{z}(\beta)},$$

where $\mathbf{z}(\beta) \equiv \beta\mathbf{x} + (1-\beta)\,\mathbf{y}$.

Proof. Choose any $\mathbf{x}$ and $\mathbf{y}$ that do not lie on the same IEP, $\alpha, \beta \in (0, 1)$, and an $\mathbf{F} \in \mathscr{L}(\mathbf{X})$. Let $\mathbf{z}(\beta) \equiv \beta\mathbf{x} + (1-\beta)\,\mathbf{y}$. Since $\succsim_{\mathbf{X}}$ are convex and monotonic we can find:

$$\begin{array}{lll} & p(\beta) \in \mathbb{R}^n_{++} & \text{such that} \quad \mathbf{z}(\beta) \in \mathbf{Y}(p(\beta)); \\ \mathbf{x}(\beta) & \text{s.t.} & p(\beta) \cdot \mathbf{x}(\beta) = p(\beta) \cdot \mathbf{x} \text{ and } \mathbf{x}(\beta) \in \mathbf{Y}(p(\beta)); \\ \mathbf{y}(\beta) & \text{s.t.} & p(\beta) \cdot \mathbf{y}(\beta) = p(\beta) \cdot \mathbf{y} \text{ and } \mathbf{y}(\beta) \in \mathbf{Y}(p(\beta)); \end{array}$$

and $\mathbf{F}_\beta \in \mathscr{L}(\mathbf{Y}(p(\beta)))$ s.t. $\mathbf{F}_\beta$ and $\mathbf{F}$ induce the same distribution over indifference sets in $\mathbf{X}$.

By Proposition 5.2, $\succsim$ satisfy AIRA for $p(\beta)$; i.e.,

$$(1-\alpha)\,\mathbf{F}_\beta \oplus \alpha\boldsymbol{\delta}_{\mathbf{z}(\beta)} \succsim (1-\alpha)\,\mathbf{F}_\beta \oplus \alpha(\beta\boldsymbol{\delta}_{\mathbf{x}(\beta)} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}(\beta)}).$$

[10] Stiglitz's result follows trivially: if $\succsim$ are income risk neutral for all prices, then preferences over commodity bundles are homothetic.

By revealed preference, since $\mathbf{x}$ and $\mathbf{y}$ do not lie on the same IEP, either $\mathbf{x} \prec \mathbf{x}(\beta)$ or $\mathbf{y} \prec \mathbf{y}(\beta)$ or both. Therefore, by ADI,

$$(1-\alpha)\,\mathbf{F}_\beta \oplus \alpha\boldsymbol{\delta}_{\mathbf{z}(\beta)} \succ (1-\alpha)\,\mathbf{F}_\beta \oplus \alpha(\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}).$$

And, by ADI again,

$$(1-\alpha)\,\mathbf{F} \oplus \alpha\boldsymbol{\delta}_{\mathbf{z}(\beta)} \succ (1-\alpha)\,\mathbf{F} \oplus \alpha(\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}). \quad \blacksquare$$

Rothblum's [12] Theorem 2 showed, among other things, that if an agent is risk neutral for all linear bifurcations then all goods are perfect substitutes for that agent; i.e., indifference curves in multi-commodity space are linear and parallel.[11] This result follows directly from the contrapositive of Proposition 5.3.

COROLLARY 5.4. *Assume* $\succsim_{\mathbf{X}}$ *strictly monotonic. If* $\succsim$ *satisfy* BRN, *then there exists* $p^* \in \mathbb{R}^n_{++}$ *s.t.* $\mathbf{Y}(p^*) = \mathbf{X}$ (*i.e., all* $\mathbf{x}$ *and* $\mathbf{y}$ *lie on the income expansion path for prices* p^*).

Proof. (by contradiction). Assume BRN but that there exists $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{x}$ and $\mathbf{y}$ do not lie on the same IEP for any p. Since BRN implies ABRA, Proposition 5.3 implies that

$$(1-\alpha)\,\mathbf{F} \oplus \alpha\boldsymbol{\delta}_{[\beta\mathbf{x}+(1-\beta)\mathbf{y}]} \succ (1-\alpha)\,\mathbf{F} \oplus \alpha(\beta\boldsymbol{\delta}_{\mathbf{x}} \oplus (1-\beta)\,\boldsymbol{\delta}_{\mathbf{y}}),$$

which is a contradiction. $\blacksquare$

APPENDIX

Proof of Proposition 3.1. The following definitions are useful. Following Chew and Epstein [2] we say that $\succsim_{\mathscr{L}(\mathbf{M})}$ exhibits monotonicity (M) if and only if $\forall \mathbf{D} \in \mathscr{L}(\mathbf{M}_p)$, $\forall m_1, m_2 \in \mathbf{M}$, $\forall \alpha \in (0, 1]$

$$m_1 > m_2 \text{ implies } \alpha\boldsymbol{\delta}_{m_1} \oplus (1-\alpha)\,\mathbf{D} \succ \alpha\boldsymbol{\delta}_{m_2} \oplus (1-\alpha)\,\mathbf{D}.$$

We say that $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ exhibits income bifurcation risk aversion (IBRA) if and only if $\forall \mathbf{D} \in \mathscr{L}(\mathbf{M}_p)$, $\forall m_1, m_2 \in \mathbf{M}$, $\forall \alpha, \beta \in [0, 1]$

$$\alpha\mathbf{D} \oplus (1-\alpha)\,\boldsymbol{\delta}_{(\beta m_1+(1-\beta)m_2)} \succsim_{\mathscr{L}(\mathbf{M}_p)} \alpha\mathbf{D} \oplus (1-\alpha)[\beta\boldsymbol{\delta}_{m_1} \oplus (1-\beta)\,\boldsymbol{\delta}_{m_2}].$$

LEMMA. $\succsim_{\mathscr{L}(\mathbf{M}_p)}$ *exihibit* M *and* IBRA *if and only if*

$$\forall \mathbf{D}, \mathbf{E} \in \mathscr{L}(\mathbf{M}_p), \qquad \mathbf{D} \text{ WSSD } \mathbf{E} \Rightarrow \mathbf{D} \succsim_{\mathscr{L}(\mathbf{M}_p)} \mathbf{E}.$$

[11] Note that, by Proposition 5.1, these $\succsim$ also satisfy Independence, thus as Safra and Segal [14, Theorem 6] show, $\succsim$ can be represented by $V(\mathbf{F}) = a \cdot (\int \mathbf{x}\, d\mathbf{F}(\mathbf{x}))$, where $a \in \mathbb{R}^n$.

This is an immediate corollary of Rothschild & Stiglitz's [13] Theorem 1(b). We know that ADI implies and is implied by $\mathbf{M}$ (see Grant, Kajii, and Polak [6] Proposition 3.2(b)), therefore it only remains to show that, given ADI, AIRA for the price system p implies and is implied by IBRA.

"$\Leftarrow$" Follows directly from the construction of AIRA.

"$\Rightarrow$" We need to show that for all $\mathbf{D}, \mathbf{E} \in \mathscr{L}(\mathbf{M}_p)$ such that $\mathbf{E}$ is a mean-preserving spread of $\mathbf{D}, \mathbf{D} \succsim_{\mathscr{L}(\mathbf{M}_p)} \mathbf{E}$. Let $\mathbf{F} \in T_p^{-1}(\mathbf{D})$ and $\mathbf{G} \in T_p^{-1}(\mathbf{E})$ such that $\mathbf{F}, \mathbf{G} \in \mathscr{L}(\mathbf{Y}(p))$. By the definition of AIRA, $\mathbf{F} \succsim \mathbf{G}$. Furthermore, by ADI, for all $\mathbf{F}' \in T_p^{-1}(\mathbf{D})$ and $\mathbf{G}' \in T_p^{-1}(\mathbf{E}), \mathbf{F}', \mathbf{G}' \in \mathscr{L}(\mathbf{X}), \mathbf{F}' \succsim \mathbf{G}'$. Hence $\mathbf{D} \succsim_{\mathscr{L}(\mathbf{M}_p)} \mathbf{E}$. ∎

Proof of Lemma 4.2. As $\mathbf{G}$ MGN $\mathbf{F}$ there exist random vectors $\mathbf{X}, \mathbf{Y}: (\Omega, \mathscr{F}, \mathbf{P}) \to \mathbf{X}$ such that the distribution of $\mathbf{X}$ (resp. $\mathbf{Y}$) is $\mathbf{F}$ (resp. $\mathbf{G}$) and $\mathbf{E}[\mathbf{Y} \mid \mathbf{X}] = \mathbf{X}$.

Fix m and choose a finite partition of $\mathbf{X}$, $\mathscr{D}_m = (\mathbf{D}_1, ..., \mathbf{D}_{K(m)})$ such that each $\mathbf{D}_k$, $k = 1, ..., K(m)$, is measurable and $\mathbf{x}, \mathbf{x}' \in \mathbf{D}_k$ implies $\|\mathbf{x} - \mathbf{x}'\| < \mathbf{M}/m$, where $\mathbf{M} = \max_{\langle \mathbf{x}, \mathbf{y} \in \mathbf{X} \rangle} \|\mathbf{x} - \mathbf{y}\|$. Note that $\mathbf{M} < \infty$ as $\mathbf{X}$ is a compact subset of $\mathbb{R}^n$. Let $\mathscr{F}_\mathbf{x}$ be the sub σ-field of $\mathscr{F}$ generated by $\{\mathbf{X}^{-1}(\mathbf{D}_k) \mid k = 1, ..., K(m)\}$, and $\mathscr{F}_\mathbf{y}$ the sub σ-field of $\mathscr{F}$ generated by $\{\mathbf{X}^{-1}(\mathbf{D}_k) \mid k = 1, ..., K(m)\} \cup \{\mathbf{Y}^{-1}(\mathbf{D}_k) \mid k = 1, ..., K(m)\}$. Note that both $\mathscr{F}_\mathbf{x}$ and $\mathscr{F}_\mathbf{y}$ are finite, and that $\mathscr{F}_\mathbf{x} \subseteq \mathscr{F}_\mathbf{y}$. Define random vectors $\mathbf{X}_m$ and $\mathbf{Y}_m$ by $\mathbf{X}_m = \mathbf{E}[\mathbf{X} \mid \mathscr{F}_\mathbf{x}]$ and $\mathbf{Y}_m = \mathbf{E}[\mathbf{Y} \mid \mathscr{F}_\mathbf{y}]$. By construction, $\mathbf{X}_m$ and $\mathbf{Y}_m$ are simple functions, hence so are their distributions. Also,

$$
\begin{aligned}
\mathbf{E}[\mathbf{Y}_m \mid \mathbf{X}_m] &= \mathbf{E}[\mathbf{Y}_m \mid \mathscr{F}_\mathbf{x}] = \mathbf{E}[\mathbf{E}[\mathbf{Y} \mid \mathscr{F}_\mathbf{y}] \mid \mathscr{F}_\mathbf{x}] \\
&= \mathbf{E}[\mathbf{Y} \mid \mathscr{F}_\mathbf{x}] = \mathbf{E}[\mathbf{E}[\mathbf{Y} \mid \mathbf{X}] \mid \mathscr{F}_\mathbf{x}] \\
&= \mathbf{E}[\mathbf{X} \mid \mathscr{F}_\mathbf{x}], \qquad \text{since} \quad \mathbf{E}[\mathbf{Y} \mid \mathbf{X}] = \mathbf{X}, \\
&= \mathbf{X}_m.
\end{aligned}
$$

Clearly by construction, $\mathbf{E}[\mathbf{X}_m] = \mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{Y}] = \mathbf{E}[\mathbf{Y}_m]$.

It remains to show that the distributions of $\mathbf{X}_m$ and $\mathbf{Y}_m$ converge to $\mathbf{X}$ and $\mathbf{Y}$, respectively. Note that for all $\omega \in \Omega$, $\|\mathbf{X}(\omega) - \mathbf{X}_m(\omega)\| < \mathbf{M}/m$ almost surely, since $\mathbf{E}[\mathbf{X} \mid \mathbf{X}(\omega) \in \mathbf{D}_k] \in \mathbf{D}_k$ almost surely. Therefore $\mathbf{X}_m$ converges to $\mathbf{X}$ in L^1 which in turn implies that the distribution of $\mathbf{X}_m$ converges to that of $\mathbf{X}$. $\mathbf{Y}_n \to \mathbf{Y}$ in distribution can be shown analogously. ∎

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